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un symposium en mémoire de Louis-André Gérard-Varet*

The Ex Ante Incentive Compatible Core in Exchange Economies With and Without Indivisibilities

Françoise Forges *

Summary

The *ex ante* incentive compatible core of an exchange economy with private information is the (standard) core of a characteristic function which expresses the fact that coalitions allocate goods by means of random incentive compatible mechanisms.

We first survey some results in the case of perfectly divisible goods. Examples then show that the *ex ante* incentive compatible core can be empty, even if utility functions are quasi-linear. If, in addition to quasi-linearity, further assumptions are made (like independent private values), the non-emptiness of the core follows nevertheless from d'Aspremont and Gérard-Varet's construction of incentive compatible, *ex post* efficient mechanisms.

We also introduce a private information version of Shapley and Scarf's economies with indivisible goods, and prove that the *ex ante* incentive compatible core is always non-empty in this framework.

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Le cœur *ex ante* incitatif d'une économie d'échange en information incomplète est le cœur, au sens de la théorie des jeux coopératifs standard, d'une fonction caractéristique qui rend compte du fait que les coalitions allouent les biens grâce à des mécanismes aléatoires incitatifs.

Nous passons d'abord en revue quelques résultats dans le cas de biens parfaitement divisibles. Des exemples montrent alors que le cœur *ex ante* incitatif peut être vide, même si les fonctions d'utilité sont quasi-linéaires. Si, en plus de la quasi-linéarité, on formule des hypothèses supplémentaires, telles que les valeurs privées et indépendantes, on peut établir la non-vacuité du cœur en construisant des mécanismes incitatifs efficaces *ex post*, suivant la méthode de d'Aspremont et Gérard-Varet.

Nous introduisons également de l'information incomplète dans les économies avec biens divisibles de Shapley et Scarf et nous démontrons que le cœur *ex ante* incitatif est toujours non-vide dans ce cadre.

Keywords: Core, incentive compatible mechanism, indivisible goods, private information.

Mots clés : Biens indivisibles, cœur, information incomplète, mécanisme incitatif.

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1. Introduction

In most collective decision problems, agents have private information on parameters entering their own utility functions as well as the others'. There is often a preliminary stage, the *ex ante* stage, at which agents do not know their precise information yet but share a common prior probability distribution on possible type profiles. At the later *interim* stage, agents learn their own types and consequently update their probability distribution over the others' types.

An *ex ante* stage typically happens when decisions are made up by representative institutions such as States. Consider for instance the application of the

“precautionary principle” to public health issues. A mechanism is typically designed at the *ex ante* stage, before individuals privately know their own state of health, and implemented at the *interim* stage (e.g., by requiring a test from the individuals). As another example, consider those States like Norway, Switzerland, etc. which are not members of the European Union but are “associated” to it. When they decide whether to participate in a particular E.U. program of scientific cooperation and how much to invest in case they participate, they do not know precisely what the quality of the applicants (laboratories, students, etc. including the national ones) will be but only have statistical data.

If an *ex ante* stage is available and binding agreements are feasible, as is assumed in classical cooperative game theory, the agents commit themselves in advance to a mechanism (i.e., a collective strategy to select decisions as functions of the agents’ types), and implement this mechanism at the *interim* stage. Since types constitute private information at this *interim* stage, one must impose incentive constraints in order to guarantee that truthful revelation be a Nash equilibrium in the game induced by the mechanism. By the well-known *revelation principle*, Bayesian incentive compatible mechanisms suffice to describe any Nash equilibrium of any non-cooperative game that the agents could design in order to exchange information and make collective decisions. There is obviously no reason to forbid mixed strategies in these general non-cooperative games so that one also has to consider *random* mechanisms – namely, lotteries selecting collective decisions as functions of the agents’ reported types.

If the underlying environment does not involve externalities, e.g., if it consists of an exchange economy with private information, incentive compatible mechanisms can be chosen not only by the group as a whole, but also by coalitions of agents. At least coalitions can pose a threat. A natural stability requirement (in addition to Pareto efficiency) is then that the grand coalition’s incentive compatible mechanism be not blocked by any smaller coalition, i.e., that no subgroup can design an incentive compatible mechanism yielding a higher expected utility to all its members. This amounts to saying that the grand coalition’s mechanism is in the *core* of a suitably defined characteristic function. We refer to this core as the “*ex ante* incentive compatible core”. In the public health example, mechanisms through which all the participants stay together make good sense, because of potential insurance effects. Natural coalitions might emerge at the *ex ante* stage: for instance, individuals who are unlikely to be infected by the underlying disease might not accept to cooperate. Solutions in the *ex ante* incentive compatible core take all these threats into account, while favoring information revelation by means of appropriate compensation schemes.

In the sequel, we focus on the *ex ante* incentive compatible core of an exchange economy with private information. In section 3, we first survey some results in standard Walrasian economies (i.e., with perfectly divisible goods and non-

transferable utility). An example in Forges, Mertens and Vohra (2002) shows that in that model, the *ex ante* incentive compatible core can be empty. However, Forges and Minelli (2001) establish a positive result when no good is initially owned by two different agents and utility functions are additively separable across goods.

We introduce particular N.T.U. economies with *indivisible* goods which extend Shapley and Scarf (1974)'s model to incomplete information. Every agent initially owns at most one indivisible item, e.g., a house, and possibly some money. Every agent has preferences over all items, which depend on the available information, but can only make use of a single item. These economies are not a strict particular case of Forges and Minelli's ones but are close to them. We give here a direct proof of the non-emptiness of the *ex ante* incentive compatible core in this model. We first show that we can, without loss of generality, restrict ourselves to "straightforward" mechanisms, whose outcomes are, for every agent, the probability of getting any given item, together with an expected amount of money. We then rely on Scarf (1967)'s theorem.

In section 4, we turn to exchange economies in which unlimited monetary transfers are allowed and utility functions are quasi-linear. As in the N.T.U. case, Forges, Mertens and Vohra (2002) provide an example of an economy with perfectly divisible goods in which the *ex ante* incentive compatible core is empty. The example is shown to be robust but features a fully informed agent and common values. As first shown by d'Aspremont and Gérard-Varet (1979) and d'Aspremont and Gérard-Varet (1982), if *interim* monetary transfers are possible, appropriate assumptions on the prior beliefs and/or the utility functions (e.g., independent private values) make first best solutions achievable in a Bayesian incentive compatible way. By relying on this result, one can establish the non-emptiness of the *ex ante* incentive compatible core in several classes of T.U. exchange economies.

We also consider unlimited monetary transfers in the economies with indivisible goods introduced in section 3. The resulting model is then an extension of Shapley and Shubik (1972)'s assignment game. Though not needed for the non-emptiness result, transfers still play an important role in this model, by making it possible to achieve first best allocations. This is illustrated by means of a detailed example.

Section 5 concludes with further observations on *interim* solution concepts.

2. Basic Definitions

An exchange economy with private information consists of

- a set of agents $N = \{1, \dots, n\}$
- l goods (each of which is either perfectly divisible or indivisible)
- for every agent $i \in N$: a finite set of types T_i , a consumption set $C_i \subseteq \mathbb{R}_+^l$, an initial endowment $e_i \in C_i$ and a von Neumann-Morgenstern utility function $u_i : T \times C_i \rightarrow \mathbb{R}$, where $T = \prod_{i \in N} T_i$. $u_i(t, \cdot)$ is assumed to be continuous and increasing for every $t \in T$.
- a probability distribution q over T , such that without loss of generality, $q(t_i) > 0$ for every $t_i \in T_i$.

Notice that consumption sets and initial endowments are independent of private information.

We assume that at the *ex ante* stage, i.e., before the agents are informed of their types, each subset of agents (or coalition) $S \subseteq N$, $S \neq \emptyset$, can decide on a mechanism μ_S , which will only be implemented at the *interim* stage (i.e., when every agent knows his type) if S has formed.

Let X_S be the set of feasible allocations:

$$X_S = \left\{ x \in \prod_{i \in S} C_i : \sum_{i \in S} x_i \leq \sum_{i \in S} e_i \right\} \quad (1)$$

and let $\Delta(X_S)$ be the set of all probability distributions over X_S . A (random¹, feasible) *mechanism* for S is a mapping $\mu_S : T \rightarrow \Delta(X_S)$ such that μ_S is measurable w.r.t. $T_S = \prod_{i \in S} T_i$, namely $\mu_S(t) = \mu_S(t')$ for every $t, t' \in T : t_S = t'_S$.

As usual, the interpretation is that every member of S has to tell his type t_i to the mechanism, which selects a feasible allocation in X_S as a function of the *reported* types. In order to define incentive compatibility, assume that agent $i \in S$, of type t_i , pretends that his type is t'_i possibly different from t_i . His expected utility from μ_S is then

$$U_i(\mu_S | t_i, t'_i) = \sum_{t_{-i}} q(t_{-i} | t_i) \int_{X_S} u_i(t_i, t_{-i}, x_i) d\mu_S(x | t'_i, t_{-i}) \quad (2)$$

Let us denote as $U_i(\mu_S | t_i)$ the (*interim*) expected utility of agent i when he truthfully reports his type to μ_S , namely

$$U_i(\mu_S | t_i) = U_i(\mu_S | t_i, t_i)$$

1. We allow for lotteries and free disposal. As shown in Forges, Minelli and Vohra (2002) and Forges, Mertens and Vohra (2002), both assumptions can be helpful in the presence of incentive constraints. Lotteries may already play a role under complete information (see, e.g., Abdulkadiroglu and Sönmez, 1988; Bogomolnaia and Moulin, 2001; Bogomolnaia and Moulin, 2002; Garrat and Qin, 1996; Hylland and Zeckhauser, 1979).

μ_S is incentive compatible (I.C.) if

$$U_i(\mu_S|t_i) \geq U_i(\mu_S|t_i, t'_i) \quad \text{for every } i \in S, t_i, t'_i \in T_i \quad (3)$$

Finally, $U_i(\mu_S)$ denotes the *ex ante* expected utility of agent $i \in S$, that is

$$U_i(\mu_S) = \sum_{t_i} q(t_i) U_i(\mu_S|t_i)$$

We associate the following (N.T.U.) characteristic function with the economy²:

$$V^*(S) = \{v \in \mathbb{R}^n : \exists \text{ an I.C. mechanism } \mu_S \text{ s.t. } U_i(\mu_S) \geq v_i \quad \forall i \in S\}$$

V^* is well-behaved: for every S , $V^*(S)$ is closed, convex (thanks to random mechanisms) and comprehensive. The *ex ante incentive compatible core* of the economy is defined as the (standard) core $C(V^*)$ of V^* , namely as the set of all vector payoffs $v \in V^*(N)$ that cannot be blocked by any coalition (i.e., there does not exist S and $w \in V^*(S)$ such that $w_i > v_i$ for every $i \in S$).

3. N.T.U. Economies

3.1. Perfectly Divisible Goods

We capture perfectly divisible goods by setting $C_i = \mathbb{R}_+^l$ for every $i \in N$ and denote the corresponding economy as $\mathcal{E}_{div} = \{N, (T_i, u_i, e_i)_{i \in N}, q\}$. In this context, Forges, Mertens and Vohra's counterexample (see Forges, Mertens and Vohra, 2002, section 6) demonstrates that there is no hope for a general result on the non-emptiness of $C(V^*)$. Nevertheless, by focusing on utility functions that are additively separable across goods, i.e., such that

$$u_i(t, x_i) = \sum_{r=1}^l u_i^r(t, x_i^r) \quad \forall i \in N, t \in T, x_i \in \mathbb{R}_+^l$$

for some continuous, increasing functions $u_i^r(t, \cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}$, Forges and Minelli (2001) established the following

Proposition 3.1 – *If in $\mathcal{E}_{div}$ no good is initially owned by two different agents (i.e., $e_i^r > 0 \Rightarrow e_j^r = 0$ for every $i, j \in N, r = 1, \dots, l$) and utility functions are additively separable across goods, the ex ante incentive compatible core is non-empty.*

Linear utility functions are obviously additively separable across goods. Furthermore, starting with arbitrary initial endowments, we can rename the goods

2. As in Holmström and Myerson (1983), “*” reminds of incentive compatibility.

according to their initial owner (i.e., good r initially owned by agent i becomes good ri) and re-express the utility functions in terms of the new goods. These utility functions are still linear. Hence

Corollary 3.1 – *If in $\mathcal{E}_{div}$ utility functions are linear, the ex ante incentive compatible core is non-empty.*

Obviously the corollary still holds with deterministic mechanisms $\mu_S : T \rightarrow X_S$; this result was identified by many authors (see Forges, Minelli and Vohra, 2002). Random mechanisms are however crucial in proposition 3.1.

3.2. Indivisible Goods

Let us turn to a particular case of the economies introduced in section 2. We set $l = k + 1$ and $K = \{1, \dots, k\}$; goods in K are indivisible items (e.g., houses) while good l is perfectly divisible (e.g., money). We assume $k \leq n$. Agent i 's consumption set is $C_i = \{0, 1\}^k \times \mathbb{R}_+$. His initial endowment $e_i \in C_i$ satisfies: $e_i^r = 1$ if $r = i \in K$, $e_i^r = 0$ if $r \in K$ but $r \neq i$, $e_i^l \geq 0$. In other words, there are k different, indivisible items; every agent $k \in K$ initially owns exactly one item; all agents may own some money.

We will focus on particular utility functions, which express that every agent only cares for a single item and that his favorite item depends on the n -tuple of types t . Let $w_i^r(t) \geq 0$ be real numbers representing the utility of item r to agent i when the types are t . We set, for $x_i \in C_i$,

$$u_i(t, x_i) = \max_{r \in K} \{w_i^r(t) I[x_i^r = 1]\} + x_i^l \quad (4)$$

where I denotes the indicator function. Observe that these utility functions are *not* additively separable across goods (except for money). We denote the economy just described as $\mathcal{E}_{ind} = \{N, K, (T_i, w_i^r(\cdot))_{i \in N, r \in K}, q\}$.

According to (1),

$$X_S = \left\{ x = \left(\begin{bmatrix} x_i^r \end{bmatrix}_{1 \leq i \leq n, 1 \leq r \leq k}, (x_i^l)_{1 \leq i \leq n} \right) \in \{0, 1\}^{n \times k} \times \mathbb{R}_+ : \begin{array}{l} \sum_{i=1}^n x_i^r \leq 1 \quad r = 1, \dots, k \\ \sum_{i=1}^n x_i^l \leq \sum_{i=1}^n e_i^l \end{array} \right\} \quad (5)$$

A feasible allocation x thus consists of an $n \times k$ matrix $\begin{bmatrix} x_i^r \end{bmatrix}$, each entry of which is 0 or 1, and an allocation of the divisible good. $x_i^r = 1$ if item r is allocated to agent i , $x_i^r = 0$ otherwise. Since there is exactly one item r in the economy, there is at most one “1” in every column r . However, at this point, several items may be allocated to the same agent.

Leaving aside incomplete information, the previous model is more general than Shapley and Scarf (1974)'s one, because it possibly involves a perfectly divisible good (in Shapley and Scarf, 1974, $e_i^l = 0$ for every i), but less general than Quinzii (1984)'s one, where the utility functions are not necessarily separable in money.

The specific form of the economy $\mathcal{E}_{ind}$ allows for a very tractable characterization of I.C. mechanisms, which are originally defined as in section 2. For simplicity, we focus on the grand coalition N . We first introduce a particular class of mechanisms, which we call *straightforward*; then we show that, without loss of generality, one can focus on straightforward mechanisms.

A straightforward mechanism associates with every $t \in T$ an $n \times k$ substochastic matrix $\pi(t)$ and a deterministic allocation $m(t)$ of the divisible good l . The first condition means that the entries of $\pi(t)$ are non-negative and that the sum of the entries in each row and each column is less than one³. More precisely, the feasibility conditions for a straightforward mechanism (π, m) are

$$\begin{aligned} \pi_i^r(t) &\geq 0, \quad m_i(t) \geq 0 \quad i = 1, \dots, n, \quad r = 1, \dots, k \\ \sum_{i=1}^n \pi_i^r(t) &\leq 1 \quad r = 1, \dots, k \\ \sum_{r=1}^k \pi_i^r(t) &\leq 1 \quad i = 1, \dots, n \\ \sum_{i=1}^n m_i(t) &\leq \sum_{i=1}^n e_i^l \end{aligned} \tag{6}$$

These inequalities are interpreted as follows: if t is the n -tuple of reported types, then agent i obtains item r with probability $\pi_i^r(t)$ and the amount $m_i(t)$ of the divisible good. The total probability of allocating item r to some agent is less than 1, and the total probability that agent i gets some item is also less than 1. By (a variant of) the Birkhoff-von Neumann theorem, any substochastic matrix is a convex combination of $n \times k$ matrices whose entries are 0 or 1, with at most one “1” in every row and every column⁴. As a consequence, every straightforward mechanism can indeed be viewed as a random, feasible mechanism in the sense of section 2, i.e., as a probability distribution over the set X_S of feasible allocations defined in (5). In addition, straightforward mechanisms never allocate two items to the same agent.

Straightforward mechanisms can also be interpreted as time-sharing schemes: $\pi_i^r(t)$ is the fraction of time during which agent i uses item r . The feasibility conditions state that no agent can use two items at the same time and that no item can be used by two agents at the same time. This interpretation allows to recover perfectly divisible goods. However, thanks to the Birkhoff-von Neumann theorem, we do not have to rely on it.

3. A “bi-stochastic” matrix corresponds to the case where these sums are exactly equal to one.

4. See Bogomolnaia and Moulin (2002) for a recent application.

Expected utilities from a straightforward mechanism (π, m) are easily computed:

$$U_i(\pi, m|t_i, t'_i) = \sum_{t_{-i}} q(t_{-i}|t_i) \left\{ \sum_{r=1}^k \pi_i^r(t'_i, t_{-i}) w_i^r(t_i, t_{-i}) + m_i(t'_i, t_{-i}) \right\}$$

Incentive compatibility of (π, m) can thus be derived as in (3).

Lemma 3.1 – Let $\mu : T \rightarrow \Delta(X_N)$ be a (random, feasible) I.C. mechanism for the grand coalition in $\mathcal{E}_{ind}$. There exists an I.C. straightforward mechanism (π, m) which achieves the same interim expected utilities as μ .

Proof: By substituting utility functions u_i from (4) into (2), we get

$$\begin{aligned} U_i(\mu|t_i, t'_i) &= \sum_{t_{-i}} q(t_{-i}|t_i) \left\{ \int_{X_S} \max_{r \in K} \{w_i^r(t) I[x_i^r = 1]\} d\mu(x|t'_i, t_{-i}) \right\} \\ &\quad + \sum_{t_{-i}} q(t_{-i}|t_i) \int_{X_S} x_i^l d\mu(x|t'_i, t_{-i}) \end{aligned} \quad (7)$$

Let us set

$$m_i(t) = \int_{X_S} x_i^l d\mu(x|t)$$

Defined in this way, m satisfies the feasibility conditions for a straightforward mechanism. Once we replace the second term in (7) as a function of m , the still relevant part of $\mu(\cdot|t)$, which appears in the first term of (7), is a probability distribution over finitely many $n \times k$ matrices $[x_i^r]$ satisfying the feasibility constraints in (5). Assume that for some t , there exists a matrix $[x_i^r]$, which we still denote as x , such that $\mu(x|t) > 0$ and with several “1” in row i . This means that, when t is reported, agent i gets several items with positive probability. Let r^* be (one of) agent i ’s favorite item(s) at t (i.e., such that $w_i^{r^*}(t) = \max_{r \in K: x_i^r = 1} w_i^r(t)$). Modify the matrix by setting $x_i^{r^*} = 1$, $x_i^r = 0$ for $r \neq r^*$ (we thus first alter the support of $\mu(\cdot|t)$ and then, if necessary, adjust the probability values). The change does not affect agents $j \neq i$ (we only considered row i), nor the expected utility of agent i at $t = (t_i, t_{-i})$. Furthermore, it may only *lower* the expected utility of agent i at (t'_i, t_{-i}) , since we have disposed of some items which did not matter for t_i but could matter for t'_i . Hence, incentive compatibility is still fulfilled. By repeating this construction, we are left with an I.C. mechanism, which we still denote as μ , selecting $n \times k$ matrices with at most one “1” in every row and in every column. (7) can be rewritten as

$$\begin{aligned} U_i(\mu|t_i, t'_i) &= \sum_{t_{-i}} q(t_{-i}|t_i) \left\{ \sum_x \sum_{r=1}^k w_i^r(t_i, t_{-i}) I[x_i^r = 1] \mu(x|t'_i, t_{-i}) \right\} \\ &\quad + \sum_{t_{-i}} q(t_{-i}|t_i) m_i(t'_i, t_{-i}) \end{aligned}$$

We can now set

$$\pi_i^r(t) = \sum_x I[x_i^r = 1] \mu(x|t)$$

(π, m) , as constructed above, defines an I.C. straightforward mechanism equivalent to μ . Q.E.D.

An analog to the previous lemma holds for every coalition S , so that the characteristic function V^* associated with $\mathcal{E}_{ind}$ is fully determined by I.C. straightforward mechanisms. Upon being measurable w.r.t. T_S , a straightforward mechanism (π_S, m_S) for coalition S will satisfy feasibility constraints similar to (6), by setting

$$\pi_{S,i}^r(t) = m_{S,i}(t) = 0 \quad \text{if } i \notin S \text{ or } r \notin K_S$$

where $K_S = \{r \in K : \exists i \in S \text{ s.t. } e_i^r = 1\}$ is the set of items initially owned by members of S . Equivalently, conditions (6) hold with “ $i \in S$ ” and “ $r \in K_S$ ” instead of “ $i \in N$ ” and “ $r \in K$ ”, respectively.

We will establish the following extension of Shapley and Scarf (1974)’s result:

Proposition 3.2 – *The ex ante incentive compatible core of $\mathcal{E}_{ind}$ is non-empty.*

As in Shapley and Scarf (1974) and Quinzii (1984), the proof applies Scarf (1967)’s theorem. We thus have to check that the game V^* induced by $\mathcal{E}_{ind}$ is balanced, i.e., that $\cap_{S \in \mathcal{S}} V^*(S) \subseteq V^*(N)$ for every balanced family of coalitions⁵. This is an immediate consequence of the following lemma:

Lemma 3.2 – Let $\mathcal{S}$ be a balanced family of coalitions with associated weights λ_S , $S \in \mathcal{S}$, and let (π_S, m_S) be a (straightforward) I.C. mechanism for S , $S \in \mathcal{S}$. Then $(\pi, m) = \sum_{S \in \mathcal{S}} \lambda_S (\pi_S, m_S)$ is a (straightforward) I.C. mechanism for N .

Proof of the lemma: Let us for instance check that $\sum_{i \in N} \pi_{ir}(t) \leq 1$. The left hand side is

$$\begin{aligned} \sum_{i \in N} \sum_{S \in \mathcal{S}} \lambda_S \pi_{S,i}^r(t) &= \sum_{i \in N} \sum_{S \in \mathcal{S}: i \in S, r \in K_S} \lambda_S \pi_{S,i}^r(t) = \sum_{S \in \mathcal{S}: r \in K_S} \lambda_S \sum_{i \in S} \pi_{S,i}^r(t) \\ &\leq \sum_{S \in \mathcal{S}: r \in K_S} \lambda_S = \sum_{S \in \mathcal{S}: a(r) \in S} \lambda_S = 1 \end{aligned}$$

where $a(r)$ denotes the initial owner of item r . The other feasibility conditions are similar. The incentive compatibility conditions are linear inequalities and thus hold for (π, m) , which is a linear combination of the (π_S, m_S) . Q.E.D.

The mechanism (π, m) in the statement of lemma 2 is not, as such, a convex combination of the (π_S, m_S) , $S \in \mathcal{S}$. Hence its feasibility had to be checked. As in the proof of proposition 3.1 (see Forges and Minelli, 2001), the previous reasoning uses the fact that each good can be identified with the agent who initially owns it.

5. Recall that a family $\mathcal{S}$ of coalitions is balanced if there are weights λ_S , $S \in \mathcal{S}$, such that $\sum_{S \in \mathcal{S}, i \in S} \lambda_S = 1$ for every $i \in N$.

However, in Forges and Minelli (2001), the grand coalition's mechanism constructed from the balanced family's mechanisms allocates the different goods *independently* of each other, which would *not* give rise to a feasible (straightforward) mechanism in the present model.

Clearly, the linearity of *interim* expected utilities $U_i(\pi, m|t_i, t'_i)$ as a function of mechanisms is extremely helpful in proposition 3.2 but this is not the only key to the result. As soon as *random* mechanisms are allowed, *interim* expected utilities *become* linear in the mechanism but this does not ensure the non-emptiness of the *ex ante* I.C. core, as shown in Forges, Mertens and Vohra (2002). If economies with random mechanisms are naively interpreted as linear, the exact feasibility condition imposed on mechanisms in section 2 is weakened to expected feasibility (see Forges and Minelli, 2001; Forges, Minelli and Vohra, 2002, section 4.3.2). In the economies with indivisible goods considered above, expected and exact feasibility are basically equivalent, thanks to the von Neumann-Birkhoff theorem. To apply this result, we needed to concentrate on straightforward mechanisms, and for this, we used the particular form of utility functions in (4). Whether proposition 3.2 holds in more general models, like the ones considered in Quinzii (1984) or Alkan and Gale (1990), is an open question.

4. T.U. Economies

Let us assume, in the basic model of section 2, that good $l = k + 1$ is perfectly divisible (let us refer to it as money) and that utility functions are quasi-linear with respect to money, i.e.,

$$u_i(t, x_i) = w_i(t, x_i^1, \dots, x_i^k) + x_i^l \quad i \in N, t \in T \quad (8)$$

where $w_i(t, \cdot)$ is continuous and increasing for every $i \in I$ and $t \in T$. Let us further assume that individual allocations in money need not be bounded below or above: $C_i = \mathbb{R}_+^k \times \mathbb{R}$ (resp., $C_i = \{0, 1\}^k \times \mathbb{R}$) if goods are perfectly divisible (resp., indivisible). Feasibility obviously requires that the aggregate allocation in money does not exceed the total initial endowment in that good. We may thus as well assume that $e_i^l = 0$ for every $i \in N$. For reasons that will be clear below, we denote the underlying economy as $\mathcal{E}_{TU}$ (we need not distinguish perfectly divisible goods from indivisible ones).

X_S can be defined exactly as in (1):

$$X_S = \left\{ x \in \prod_{i \in S} C_i : \sum_{i \in S} x_i^r \leq \sum_{i \in S} e_i^r \quad r = 1, \dots, k, \quad \sum_{i \in S} x_i^l \leq 0 \right\}$$

According to the definition of section 2, a mechanism μ_S for coalition S is described by probability distributions over X_S for every type profile t . However,

since the utility functions are quasi-linear, the incentive compatibility constraints only depend on the conditional expected monetary transfers. Hence we can restrict ourselves without loss of generality to *deterministic* money transfers and describe μ_S as (ξ_S, m_S) , where ξ_S selects feasible allocations of goods $r = 1, \dots, k$ and $m_S : T \rightarrow \mathbb{R}^N$ selects feasible money transfers, namely $\sum_{i \in S} m_{S,i}(t) \leq 0$.

Furthermore, type independent monetary transfers are obviously incentive compatible, so that the characteristic function V^* defined in section 2 reduces to the following T.U. characteristic function:

$$v^*(S) = \max_{(\xi_S, m_S) \text{ I.C.}} \sum_{i \in S} U_i(\xi_S, m_S)$$

The *ex ante* incentive compatible core simplifies into the set $C(v^*)$ of all vector payoffs $v \in \mathbb{R}^N$ such that $\sum_{i \in N} v_i \leq v^*(N)$ and, for every S , $\sum_{i \in S} v_i \geq v^*(S)$.

For comparison, we also define the characteristic function v which prevails in the absence of incentive constraints. In this case, free disposal of money cannot be useful. Moreover, if the utility functions are concave, lotteries are not necessary either. Hence v takes the simple form:

$$v(S) = \max_{\xi_S \text{ deterministic}} \sum_{i \in S} \sum_{t \in T} q(t) u_i(t, \xi_{S,i}(t))$$

By the complete information results (see Hildenbrand and Kirman, 1988, for perfectly divisible goods and Shapley and Shubik, 1972, for indivisible goods), $C(v)$ is not empty. If $v^*(N) = v(N)$, namely if incentive compatibility constraints do not matter in the grand coalition, $C(v) \subseteq C(v^*)$, so that the *ex ante* incentive compatible core is also non-empty.

Starting with d'Aspremont and Gérard-Varet (1979 and 1982), a number of papers (e.g., d'Aspremont, Crémer and Gérard-Varet, 1990; d'Aspremont, Crémer and Gérard-Varet, 2002; Crémer and McLean, 1985; Crémer and McLean, 1988; Johnson, Pratt and Zeckhauser, 1990) have identified conditions on the beliefs q and/or the utility functions $w_i(t, x_i^1, \dots, x_i^k)$ which guarantee that the grand coalition N can achieve first best allocations by means of an incentive compatible mechanism with budget balanced transfers⁶, which implies that $v^*(N) = v(N)$. Hence these conditions also ensure the non-emptiness of $C(v^*)$. We illustrate this kind of result with two particular simple sets of sufficient conditions (see Forges, Mertens and Vohra, 2002, for further details and other results making use of the same reasoning).

Proposition 4.1 – *If in $\mathcal{E}_{TU}$ the beliefs are independent (i.e., $q(t) = \prod_{i \in N} q(t_i)$ for every t) and values are private (i.e., $w_i(t, (x_i^r)_{1 \leq r \leq k}) = w_i(t_i, (x_i^r)_{1 \leq r \leq k})$ for every $i, t, (x_i^r)_{1 \leq r \leq k}$), the ex ante incentive compatible core is non-empty.*

6. Namely, (ξ, m) such that $\sum_{i \in N} m_i(t) = 0$.

The next proposition is established in Johnson, Pratt and Zeckhauser (1990). Let us denote as $q_{i,i+1}(\cdot|t_i)$ agent i 's conditional probability distribution (induced by q) over T_{i+1} given t_i , where $i+1$ is understood *mod. n*.

Proposition 4.2 – *If in $\mathcal{E}_{TU}$ the beliefs satisfy*

$$q_{i,i+1}(\cdot|t_i) \neq q_{i,i+1}(\cdot|s_i) \quad \forall i, \forall t_i, s_i, t_i \neq s_i,$$

the ex ante incentive compatible core is non-empty.

The previous analysis applies in economies with perfectly divisible goods as well as in the T.U. version of the economies with indivisible goods that we considered in section 3.2. However, monetary transfers do not play any essential role to make the core non-empty in the latter case, since it was already non-empty with non-transferable utilities. More precisely, proposition 3.2 holds in the T.U. case, and the proof of the analog of lemma 2 can be simplified by relying on the Bondareva-Shapley theorem instead of Scarf, 1967's (see Forges, 2004). Although unlimited monetary transfers are not necessary for the non-emptiness of the core, they are nevertheless useful to achieve first best solutions in the *ex ante* incentive compatible core. This is illustrated in the next example.

Example

Let us assume that $n = 4$ and $k = 2$. Agents 1 and 2, who can be viewed as sellers, both own an item, say, a house, and know, at the *interim* stage, whether the quality of their own house is high or low: $T_1 = \{h_1, l_1\}$, $T_2 = \{h_2, l_2\}$. The probability distribution over the sellers' types is $q(h_1, h_2) = q(l_1, l_2) = \frac{3}{8}$, $q(h_1, l_2) = q(l_1, h_2) = \frac{1}{8}$. Agents 3 and 4, the potential buyers, have no private information. The reservation price of the sellers is p_h (resp., p_l) for a high (resp., low) quality house. In other words, the utility from selling his house is $-p_h$ (resp., $-p_l$) for a high (resp., low) type seller. The reservation prices of the buyers are u_h and u_l respectively, i.e., acquiring a high (resp., low) quality house yields utility u_h (resp., u_l) to each buyer. We assume that $u_l < p_l < p_h < u_h$ and that $\frac{1}{2}(u_l + u_h) < \frac{1}{2}(p_l + p_h)$ (e.g., 0, 9, 12, 20) and that the utility of any agent is 0 when no sale takes place.⁷

Since the buyers are not submitted to incentive constraints, we will, without loss of generality, focus on transfers summing up to 0 throughout the example.

Let us first consider a seller-buyer coalition $\{i, j\}$, $i \in \{1, 2\}$, $j \in \{3, 4\}$. The seller has then two equiprobable types, which we denote as h and l . We face a simple,

7. For a concrete example, think of the termites invading some regions of France. The qualities of houses in the same neighborhood are highly correlated. We assume that mechanisms are designed before the sellers know the quality of their houses, which is consistent with the fact that quality is only checked in case of potential transactions.

discrete version of Myerson, 1985's "lemon problem". A mechanism for a seller-buyer coalition consists of the probability of trade π_h (resp., π_l) when the seller reports type h (resp., l) and the corresponding expected transfers m_h, m_l from the buyer to the seller (note that the transfers include the sale price as well as possible fees). By eliminating the transfers from the incentive constraints (see, e.g., Johnson, Pratt and Zeckhauser, 1990 or Myerson, 1985), the optimization problem of a seller-buyer coalition $\{i, j\}$, $i \in \{1, 2\}, j \in \{3, 4\}$ is

$$v^*({i,j}) = \max \left[\frac{1}{2} \pi_h (u_h - p_h) + \frac{1}{2} \pi_l (u_l - p_l) \right] \quad s.t. \quad 0 \leq \pi_h \leq \pi_l \leq 1$$

so that, under our assumptions,

$$v^*({i,j}) = 0 \quad i \in \{1, 2\}, j \in \{3, 4\}$$

Let us set $g_h = u_h - p_h > 0$. Observe that, in absence of incentive constraints, $v({i,j}) = \frac{1}{2} g_h$. Trade is indeed beneficial in state h , but the incentive compatibility conditions prevent revelation of information from the seller.

Let us turn to the grand coalition $N = \{1, 2, 3, 4\}$. First best efficiency requires to sell the high quality houses, and only those, at every state of nature. Hence,

$$v(N) = g_h$$

We will construct an incentive compatible mechanism achieving g_h as sum of expected payoffs, so that

$$v^*(N) = g_h$$

As observed above, this implies that $C(v) \subseteq C(v^*)$ and provides a simple procedure to construct expected payoffs in the *ex ante* incentive compatible core.⁸

Consider a mechanism in which only high quality houses are sold, namely

$$\begin{aligned} \pi(h_1, h_2) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \pi(h_1, l_2) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \pi(l_1, h_2) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \pi(l_1, l_2) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

The sum of expected payoffs from π is g_h . Obviously, π is not incentive compatible, but one can construct transfers m such that (π, m) is incentive compatible. For instance, the transfers m_1 to the first seller must satisfy

$$\begin{aligned} -p_h + \frac{3}{4} m_1(h_1, h_2) + \frac{1}{4} m_1(h_1, l_2) &\geq \frac{3}{4} m_1(l_1, h_2) + \frac{1}{4} m_1(l_1, l_2) \\ \frac{1}{4} m_1(l_1, h_2) + \frac{3}{4} m_1(l_1, l_2) &\geq -p_l + \frac{1}{4} m_1(h_1, h_2) + \frac{3}{4} m_1(h_1, l_2) \end{aligned}$$

8. The procedure can be applied to a large class of mechanisms (see, e.g., d'Aspremont, Crémer and Gérard-Varet, 1990; d'Aspremont, Crémer and Gérard-Varet, 2002; d'Aspremont and Gérard-Varet, 1979; d'Aspremont and Gérard-Varet, 1982; Crémer and McLean, 1985; Crémer and McLean, 1988; Johnson, Pratt and Zeckhauser, 1990).

A possible solution is

$$\begin{aligned} m_1(h_1, h_2) &= \frac{3p_h - p_l}{2} + \frac{g_h}{4} & m_1(h_1, l_2) &= \frac{3p_l - p_h}{2} + \frac{g_h}{4} \\ m_1(l_1, h_2) &= \frac{g_h}{4} & m_1(l_1, l_2) &= \frac{g_h}{4} \end{aligned}$$

The transfers m_2 to the second seller can be chosen in a similar way. In order to balance the transfers, one can simply set $m_3 = -m_1$, $m_4 = -m_2$. The mechanism (π, m) thus associates buyer 3 (resp., 4) with seller 1 (resp., 2) but sale only takes place if the seller's house is of high quality. (π, m) yields the expected payoff $\frac{g_h}{4}$ to each trader. The mechanism reflects that sale prices are influenced by the presence of low quality items; the transfers in the low state should be interpreted as a fee that the potential buyers pay to get information and avoid a bad decision. Many other mechanisms achieving ex post efficiency can be constructed. In particular, as in Crémer and McLean (1988), it is possible to design the transfers in such a way that the mechanism is *interim* individually rational for the sellers, who fully extract the surplus⁹.

Let us end the analysis of the example by showing that the expected payoff from (π, m) , namely, $(\frac{g_h}{4}, \frac{g_h}{4}, \frac{g_h}{4}, \frac{g_h}{4})$, belongs to $C(v^*)$. We have evaluated $v(\{i, j\})$, $i \in \{1, 2\}$, $j \in \{3, 4\}$ and $v(N)$. To complete the description of v , we compute that

$$\begin{aligned} v(\{i, 3, 4\}) &= \frac{g_h}{2} & i &= 1, 2 \\ v(\{1, 2, j\}) &= \frac{5g_h}{8} & j &= 3, 4 \end{aligned}$$

Hence, $(\frac{g_h}{4}, \frac{g_h}{4}, \frac{g_h}{4}, \frac{g_h}{4}) \in C(v)$. It follows from our previous remarks that $(\frac{g_h}{4}, \frac{g_h}{4}, \frac{g_h}{4}, \frac{g_h}{4}) \in C(v^*)$. The same reasoning applies to $(\frac{g_h}{2}, \frac{g_h}{2}, 0, 0)$.

This example shows that it may be better for the agents (in the sense of generating a higher sum of expected payoffs) to stay together at the *ex ante* stage in order to exchange information *within the grand coalition*. For instance, this enables the agents to exploit the possible correlation between types and to achieve first best efficiency through full revelation.

9. Adding (resp., subtracting) $\frac{g_h}{4}$ to (resp., from) all previous transfers gives the surplus to the sellers (resp., buyers). All these mechanisms, including the latter, are interim individually rational for the sellers. Another mechanism with the same properties is

$$\begin{aligned} m_1(h_1, h_2) &= \frac{3u_h - p_l}{2} & m_1(h_1, l_2) &= \frac{3p_l - u_h}{2} \\ m_1(l_1, h_2) &= \frac{3g_h}{2} & m_1(l_1, l_2) &= \frac{-g_h}{2} \end{aligned}$$

5. Concluding Remarks

In this paper, we have focused on the *ex ante* incentive compatible core, which is defined without any ambiguity, as the classical core of a well-behaved characteristic function. As pointed out by Myerson (see, e.g., Myerson, 1983; Myerson, 1984; Myerson, 1995; Myerson, 2002), it might be that no *ex ante* stage is available. In many examples, coalitions do form at the *interim* stage. The main issue then is that the negotiation over mechanisms already conveys information on the agents' types, so that a completely different approach is needed. This problem is still widely open (see de Clippel, Geoffroy and Minelli, 2004; Dutta and Vohra, 2005; Myerson, 1995; Myerson, 2002, for a discussion of the issues and possible solutions; Forges, Minelli and Vohra, 2002, surveys most available results).

The previous difficulties disappear if coalitions *form* at the *interim* stage, but cannot exchange information at that stage. This approach was followed by Wilson (1978) and Vohra (1999) in order to define the *coarse core* and the *incentive compatible coarse core*, respectively. The latter solution concept assumes that coalitions S use incentive compatible mechanisms μ_S as in this paper. But coalition S can block proposals from the grand coalition at the *interim* stage, namely as soon as it is common knowledge in S , at that stage, that some incentive compatible mechanism improves the expected payoffs of all members of S (i.e., there exists an incentive compatible mechanism yielding a higher payoff to *all types* in T_S that are common knowledge in S). As a consequence, mechanisms in the incentive compatible coarse core are *interim* individually rational (see, e.g., Crémer and McLean, 1985; Crémer and McLean, 1988) and *interim* incentive efficient (in the sense of Holmström and Myerson, 1983).

The incentive compatible coarse core is empty in Forges, Mertens and Vohra (2002)'s example, both in the T.U and the N.T.U. case. It is non-empty in exchange economies with linear utility functions or with divisible goods as in section 3.2: both corollary 3.1 and proposition 3.2 apply to the incentive compatible coarse core (see Forges, 2004, for a detailed proof of the result in the case of T.U. economies with indivisible goods). This might let us hope that the incentive compatible coarse core behaves in the same way as the *ex ante* incentive compatible core. However, nothing of the kind is clear for the results which depend on unlimited monetary transfers, as propositions 3, 4 and 5. Even the basic T.U. structure of the model disappears at the *interim* stage (see Forges, Minelli and Vohra, 2002; Forges, Mertens and Vohra, 2002, for further comments).

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