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LEARNING FROM AND ADAPTING THE THEORY OF REALISTIC MATHEMATICS EDUCATION

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Abstract: This article focuses on the critical role of design theory in our work as mathematics educators. We give particular attention to a specific design theory, Realistic Mathematics Education (RME). We first clarify the enduring contributions of RME to design in mathematics education and then discuss three adaptations that we made to RME theory while conducting a series of classroom design experiments. The first of these adaptations involves taking a broader perspective on the means of supporting students' mathematical learning to include both the organization of classroom activities and the nature of classroom discourse. The second adaptation involves a change in orientation that acknowledges the mediating role of the teacher. The goal of instructional design then becomes to develop resources that teachers can use to achieve their instructional agendas rather than to support students' learning directly. The third adaptation again centers on the teacher and concerns the potential contribution of designed instructional resources as a means of supporting teachers' as well as students' learning.

Keywords: design research, instructional design, design theory, Realistic Mathematics Education, mathematics teaching and learning.

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Our focus in this article is on the critical role that design theory has played in our work as mathematics educators. To ground the discussion, we give particular attention to a specific design theory that we found especially useful in our work over the past 15 years, Realistic Mathematics Education (RME) developed at the Freudenthal Institute in the Netherlands. We first outline the type of research in which we engaged when we first began to appropriate ideas from RME design theory. Against this background, we clarify the significance that we attributed to RME theory at that time and then move to the present in order to discuss our current views of the contributions of this design theory. In the second part of the paper, we focus on three adaptations we have made to RME theory as our theoretical position and research interests have evolved over time. The first of these adaptations involves a broadening of our perspective on the means of supporting students' mathematical learning to include both the organization of classroom activities and the nature of the classroom discourse. The

second adaptation involves a shift in our orientation as instructional designers such that our goal when developing sequences of instructional activities is no longer to support individual students' learning directly. We now consider the mediating role of the teacher as crucial and view ourselves as developing resources for teachers to use to achieve their instructional agendas. The third adaptation again centers on the teacher and concerns the potential contribution of instructional sequences as a means of supporting teachers' as well as students' learning.

The Contributions of RME

RME is rooted in Freudenthal's (1971; 1973) interpretation of mathematics as a human activity. In Freudenthal's view, students should be given the opportunity to reinvent mathematics by organizing or mathematizing either real world situations or mathematical relationships and processes that have substance for them. In developing this position, Freudenthal emphasized that the material students are to mathematize should be real for them. It is for this reason that approach is called *Realistic Mathematics Education*. Freudenthal considered

mathematizing to be the key process in mathematics education for three reasons. First, mathematizing is a major activity of mathematicians. Second, mathematizing fosters applicability by familiarizing students with a mathematical approach to everyday settings. Third, mathematizing relates directly to the idea of reinvention, a process in which students formalize their informal understandings and intuitions. Freudenthal argued with considerable force that mature, conventional symbolizations should not be taken as the instructional starting point. He was particularly critical of this practice and termed it an anti-didactic inversion because the process by which the mathematicians developed mathematics is turned upside down (Freudenthal, 1973). For Freudenthal and for researchers and instructional developers who subsequently elaborated his ideas, the goal of mathematics education should be to support students' mathematical learning as a process of guided reinvention.

Positive Heuristics for Instructional Design

To provide a context for the subsequent discussion, we first outline the type of research we were conducting at the time that we became aware of RME and began to focus explicitly on issues of instructional design. This research involved conducting a series of year-long design experiments in US second-grade and third-grade classrooms with seven- and eight-year-old children. One of our goals in conducting these experiments was to extend the constructivist teaching experiment methodology developed by Steffe and his colleagues to the classroom (Cobb & Steffe, 1983; Steffe, 1983; Steffe & Kieren, 1994; Steffe & Thompson, 2000). In a constructivist teaching experiment, the researcher typically interacts with a small number of students one-on-one and attempts to precipitate their learning by posing judiciously chosen tasks and by asking follow up questions, often with the intention of encouraging the student to reflect on his or her mathematical activity. The immediate purpose when interacting with the participating students is to study the *process* by which they reorganize their mathematical reasoning. The larger purpose when conducting a retrospective analysis of the teaching sessions is to develop conceptual models composed of theoretical constructs that can be used to account for the mathematical learning

of other students (P. W. Thompson & Saldanha, 2000). As a consequence, although the researcher acts as a teacher when conducting an experiment of this type, the primary emphasis is psychological and centers on the analysis of students' mathematical reasoning. The development of instructional designs is, in contrast, of secondary importance.

Our primary focus when we extended this methodology to the classroom remained consistent with Steffe's emphasis on developing explanatory theoretical constructs rather than on formulating instructional designs. In particular, we gave priority to the development of an interpretive framework that would enable us to situate students' mathematical learning within the social context of the classroom (cf. Cobb & Yackel, 1996). This was the case even though we had developed a complete set of instructional activities during a design experiment conducted in a second-grade classroom. Constructivism and the related theories on which we drew provided us with a number of negative heuristics that ruled out a range of approaches to instructional design. For example, we rejected what Treffers (1987) termed structuralist approaches to design in which physical materials and graphics are developed that, for the adult, embody the mathematical relationships that are the target of instruction. However, the positive heuristics that guided our development of instructional activities were global in nature. For example, the constructivist perspective to which we subscribed at that time oriented us to view mathematical learning as an activity in which students reorganize their activity to resolve situations that they experience as problematic. The instructional activities that we developed were therefore designed to give rise to a range of mathematical problems as students interpreted the activities at a variety of levels of sophistication. We also drew on neo-Piagetian analyses of social interactions (Doise & Mugny, 1979) and viewed classroom social interactions as a potential means of supporting students' progressive reorganization of their mathematical reasoning. In particular, we conjectured that interactions in which conflicts in students' interpretations became apparent would give rise to learning opportunities for them, and that they might reorganize their reasoning as they resolved these interpersonal conflicts. As a consequence, the structure of classroom activities in the second- and third-grade design experiments involved the students

first working in pairs to complete instructional activities and then participating in whole-class discussions of their interpretations and solutions.

It was not until we first began to learn about RME theory that we came to realize that our attempts at instructional design were, in comparison, inadequate and that we needed to develop a more systematic approach. Our interest in RME design theory was first inspired by Adrian Treffers's (1987) book, *Three dimensions: A model of goal and theory description in mathematics instruction - The Wiskobas Project*. Treffers's intent in this book was primarily theoretical in that he schematized two decades of instructional design work and classroom experimentation, in the process teasing out a number of positive heuristics for instructional design in mathematics. Against the background of our research at the time, the design heuristics that Treffers illustrated by describing several instructional sequences were a revelation. It was apparent from Treffers's account that RME was a detailed, empirically grounded design theory that was compatible with our constructivist perspective on mathematical learning. For example, RME's basic tenets of mathematics as a human activity and of mathematical learning as the progressive reorganization of activity were both consistent with our general viewpoint. In addition, we valued the manner in which RME placed students' mathematical reasoning at the center of the design process while simultaneously proposing the specific means by which the development of their reasoning could be systemically supported. Furthermore, Treffers's account of RME clarified that the purpose for conducting design experiments was not limited to developing explanatory constructs, but could also include developing, testing, and revising instructional sequences.

These insights led us to realize that we had attempted to study students' mathematical learning in situations in which we had not necessarily provided adequate supports for learning. The design decisions that Treffers illustrated served to emphasize a relatively fine-grained level of detail that had been absent in our prior work. The types of design decisions that we especially noted included both the careful selection of the problem situations that were used during the first part of an instructional sequence and the explicit attention given to

the design of non-standard notation schemes as a means of supporting students' reorganization of their mathematical activity. In addition to taking account of specific aspects of RME theory, we also learned an important methodological lesson from Treffers's presentation. The heuristics on which we had relied were relatively global in nature because we had derived them from a general background theory. In contrast, the specific design heuristics that Treffers outlined had emerged from and yet remained grounded in the activities of designing and experimenting in classrooms.

As these reflections make clear, there was much that we could learn by becoming familiar with the general orientation to design inherent in RME. However, we were also aware that RME cannot be reduced to a set of heuristics. It is a sophisticated set of practices that have been developed by a community of designers and researchers over an extended period of time. Gravemeijer (1994) clarifies why the process of developing and revising designs in the RME tradition cannot, in principle, be completely codified. In reflecting on his activity as an instructional designer, Gravemeijer argues that designer's activity resembles that of a bricoleur:

A bricoleur is a handy man who invents pragmatic solutions in practical situations...[T]he bricoleur has become adept at using whatever is available. The bricoleur's tools and materials are very heterogeneous: Some remain from earlier jobs, others have been collected with a certain project in mind. (p. 447)

From the sociocultural perspective, this is an excellent characterization of creativity and invention (cf. Cole, 1996). Novelty and originality are seen to be historically and culturally situated from this perspective and to involve the adaptation and reconfiguring of resources from a range of disparate practices (cf. Emirbayer & Mische, 1998; Holland, Skinner, Lachicotte, & Cain, 1998; Varenne & McDermott, 1998). Inspired by Gravemeijer's characterization of design, we have in fact described our own efforts to develop theoretical constructs that enable us to situate students' mathematical learning within the social context of the classroom as a process of bricolage in which we drew on and adapted ideas from a range of theoretical sources for pragmatic ends (Cobb, Stephan, McClain, & Gravemeijer, 2001).

This portrayal of RME as an evolving set of interrelated practices implies that it is one thing to attempt to understand a particular approach to design by reading about it and quite another to engage competently in the process of developing designs. As a consequence of reading books by leading contributors to RME (e.g., Gravemeijer, 1994; Streefland, 1991; Treffers, 1987), we became reasonably proficient at commentating on the design theory. However, we had to co-participate in the process of formulating, testing, and revising designs before we became able to develop adequate designs for supporting students' mathematical learning. In this regard, an ongoing collaboration with Koeno Gravemeijer in the course of which we developed several instructional sequences proved to be crucial. With hindsight, we view our collaboration with Gravemeijer as a process of apprenticeship in which we were legitimate peripheral participants in the RME research community (cf. Lave & Wenger, 1991). In the course of this apprenticeship, we consciously adapted several of the ideas that we appropriated from RME while pursuing a number of other interests. In the remainder of this article, we first step back from our personal involvement in RME to consider its enduring contributions to design in mathematics education and then discuss three of these adaptations.

Enduring Contributions

In our view, RME theory makes an enduring contribution to mathematics education by offering a viable solution to a long-standing problem that was first delineated by John Dewey. In contemporary accounts, Dewey is frequently characterized as a strong advocate of open-ended, project-based approaches to instruction. It is true that Dewey condemned the predominant instructional practice of his day that promoted rote learning at the expense of a deep understanding of disciplinary ideas. However, as Prawat (1995) documents, Dewey was equally critical of unstructured project-based approaches in which students are encouraged to pursue their own interests (Westbrook, 1991). He argued that these two general instructional approaches constitute opposite poles of a dichotomy between disciplinary content on the one hand and students' current understandings and interests on the other. A primary goal of both his educational philosophy and his work

at the Laboratory School that he established at the University of Chicago was to transcend this dichotomy. In this regard, he anticipated a central issue with which mathematics educators continue to struggle. Ball (1993) articulated this issue succinctly when she asked:

How do I [as a mathematics teacher] create experiences for my students that connect with what they now know and care about but that also transcend the present? How do I value their interests and also connect them to ideas and traditions growing out of centuries of mathematical exploration and invention? (p. 375, italics in the original)

The originality of the solution proposed by RME stems from the manner in which this perennial problem is reframed. For RME designers, the challenge is not that of directly connecting students' understandings and interests with the ideas of the discipline. Instead, it is to support the progressive development of students' mathematical reasoning so that they can eventually participate in established mathematical practices that have grown out of centuries of exploration and invention. The metaphor inherent in the RME approach is that of continually *building up towards* substantial participation in established mathematical practices rather than that of attempting to directly *connect or bridge between* students' current understandings and the established mathematical ideas and traditions that constitute students' intellectual inheritance.

This orientation has become routine for RME designers and is captured by three central tenets of the design theory. The first of these tenets is that the starting points of an instructional sequence should be *experientially real* to students in the sense that they can immediately engage in personally meaningful mathematical activity (Gravemeijer, 1990; Streefland, 1991). For the designer, the immediate goal is that students' interpretations and solutions should lead to the development of informal ways of speaking, symbolizing, and reasoning across a range of initial instructional activities. Treffers (1987) calls this process of negotiating and generalizing informal solution processes horizontal mathematization. Thompson (1992) indicates the significance of this first tenet from a constructivist perspective when he observes that "if students do not become engaged

imaginistically in the ways that relate mathematical reasoning to principled experience, then we have little reason to believe that they will come to see their worlds outside of school as in any way mathematical" (p. 10). This tenet is also consistent with recommendations derived from investigations that have compared and contrasted mathematical activity in the classroom with that in out-of-school situations (e.g., de Abreu, 2000; Masingila, 1994; Nunes, Schliemann, & Carraher, 1993; Saxe, 1994).

The second central tenet is that in addition to being experientially real to students, the starting points should also be justifiable in terms of the potential end points of the learning sequence. This implies that the informal ways of speaking, symbolizing, and reasoning established during the initial phase of an instructional sequence should constitute a basis for a progressive process of vertical mathematization. However, as Treffers makes clear, students' increasingly sophisticated mathematical reasoning does not become decoupled from the starting point situations in this process. Instead, these situations should continue to function as paradigm cases to which students can fold back (Pirie & Kieren, 1994), thereby anchoring their reasoning. This latter requirement is consistent with analyses that emphasize the important role that analogies (Clement & Brown, 1989) and metaphors (Dörfler, 2000; Presmeg, 1997) play in mathematical activity.

The third tenet focuses on the means of supporting the process of vertical mathematization and it is here that the general approach of building up towards substantial participation in established mathematical practices becomes evident. One of the primary means of support involves activities in which students create and elaborate symbolic models of their informal mathematical activity. This modeling activity might involve making drawings, diagrams, or tables, or it could involve developing informal notations or using conventional mathematical notations. This third tenet is based on the conjecture that, with the teacher's guidance, students' *models of* their informal mathematical activity can evolve through use into *models for* more general mathematical reasoning (Gravemeijer, 1999; Gravemeijer, Cobb, Bowers, & Whitenack, 2000). This transition involves a shift such that ways of symbolizing initially developed to mathematize informal activity subsequently come

to support more general mathematical activity in a range of situations. In practice, the approach of building up from students' initial informal activities to conventional ways of symbolizing involves both the judicious selection of instructional activities and the negotiation of successive ways of symbolizing. As part of this process, the teacher attempts to achieve his or her instructional agenda by capitalizing on students' contributions and by introducing ways of symbolizing that fit with their reasoning at particular points in an instructional sequence.

This third tenet of RME is consistent with Sfard's (1991; 1994) historical analysis of several mathematical concepts including number and function (cf. Gravemeijer et al., 2000). Sfard contends that the historical development of mathematics can be seen as a long sequence of reifications, each of which involves the transformation of operational or process conceptions into object-like structural conceptions. She argues that the development of ways of symbolizing has been integral to the reification process. However, she is also careful to clarify that it is the process of reasoning with symbolizations, not the symbolizations themselves, that are reified during a model-of to model-for transition. This emphasis on activity serves to differentiate approaches based on RME from approaches that involve a so-called modeling point of view (e.g., Lesh & Doerr, 2000). In these latter approaches, a model is considered to capture mathematical structures or relationships inherent in starting point situations. In contrast, models as they are characterized in RME originate from students' ways of acting and reasoning with tools and symbols in the starting point situations. Although this distinction is subtle, it has important implications for design in that the focus is on students' anticipated interpretations and solutions rather than on the features of instructional activities per se.

The attention that RME designers give to the development of ways of symbolizing suggests a possible point of contact with Vygotsky's cultural-historical theory of development. Semiotic mediation and the use of cultural tools such as mathematical symbols constitutes one of the two mechanisms that Vygotsky contended drives conceptual development, the other being interpersonal relations (Davydov, 1995; van der Veer & Valsiner, 1991; Vygotsky, 1987). The differences between RME and Vygotskian appra-

ches center on the contrast between the metaphors of building up towards substantial participation in established mathematical practices and of connecting students' current understandings with established mathematical ideas. In Vygotskian theory, learning in instructional situations is viewed as a process of transmitting mathematical meaning from one generation to the next. Within this perspective, symbols are sometimes called carriers of meaning and are treated as primary vehicles of the enculturation process (van Oers, 1996). This formulation does not, of course, imply a crude transmission view of communication. Instead, the fundamental claim is that students develop particular mathematical conceptions as they learn to use conventional symbols while engaging in particular sociocultural activities (Davydov, 1988). In this scheme, the teacher's role is frequently characterized as that of relating students' personal meanings to the cultural meanings inherent in the appropriate use of conventional symbols. The teacher's role might therefore be characterized as that of introducing conventional means of symbolizing and relating them to students' mathematical activity (cf. Davydov & Radzikhovskii, 1985; Tharp & Gallimore, 1988).

In contrast to the sociocultural framing of instructional design as the transmission of mathematical meaning from one generation to the next, RME designers frame the fundamental design challenge as that of supporting the emergence of mathematical meaning in the classroom. This orientation is apparent in both the means of symbolizing developed in the classroom and in the characterization of the teacher's role. As is the case in Vygotskian approaches, RME acknowledges that the teacher is an institutionalized authority in the classroom. Further, the teacher might, on occasion, express this authority in action by introducing means of symbolizing as he or she redescribes students' contributions. However, it is apparent from the third tenet of RME that these means of symbolizing are not restricted to conventional mathematical symbols. Instead, the designer draws on both historical analyses and analyses of students' informal mathematical reasoning to invent means of symbolizing that students, at a particular point in their development, might see as reasonable to use to achieve their mathematical goals (Gravemeijer, 1994). The resulting instructional sequences therefore involve the establishment of non-standard means of symbolizing that are designed both to fit

with students' informal activity and to support their development of more sophisticated forms of mathematical reasoning. These means are, in effect, offered to students as resources that they might use as they solve problems and communicate their thinking. In this approach to design, symbolic means are developed to support students' progressive reconstruction of cultural meanings (Gravemeijer, 1994).

Adaptations

We now consider three adaptations we have made that build on the RME approach to design. In doing so, we draw on a recently completed classroom design experiment that focused on the teaching and learning of statistical data analysis at the middle school level. We give an overview of the design experiment as we discuss the first adaptation that involves broadening our perspective on the means of supporting students' mathematical learning. As we illustrate, this broader perspective enables us to view classrooms as activity systems that are designed to support the participating students' learning of significant mathematical ideas.

Classrooms as Activity Systems

Background to the design experiment

The design experiment on which we will focus was conducted in a US seventh-grade classroom with 29 12-year-old students and focused on the analysis of univariate data. The experiment, which was conducted in collaboration with Koeno Gravemeijer and Erna Yackel, lasted 12 weeks and involved 34 classroom sessions of approximately 40 minutes in duration. A member of the research team¹ served as the teacher throughout the experiment. As part of the process of preparing for the experiment, we identified both the instructional starting points and prospective endpoints. The interviews and whole class performance assessments that we conducted to determine the starting points indicated that, for most of the students, data analysis involved "doing something with the numbers" by manipulating them in a relatively procedural manner (McGatha, Cobb, & McClain, 2002). For example, many of the students simply calculated mean of every data

set when attempting to solve each task posed irrespective of whether the mean would give them useful insights into the problem. Our analysis of these interviews indicated that the students did not view data as measures of aspects or features of a situation that had been generated in order to understand a phenomenon or make a decision. We concluded from these assessments that our immediate goal should be to influence the students' beliefs about what it means to do statistics in school such that they would begin to analyze data in order to address a significant question rather than simply perform calculations and follow conventions for drawing specific types of graphs.

The identification of prospective instructional endpoints involved delineating what Wiggins and McTighe (1998) term the "big ideas" that are at the heart of a discipline, that have enduring value beyond the classroom, and that offer potential for engaging students. The overarching statistical idea that emerged from our synthesis of the research literature and our analysis of the interviews and classroom performance assessments was that of *distribution*. One of primary goals for the design experiment was therefore that the students would come to view data sets as entities that are distributed within a space of possible values (Hancock, Kaput, & Goldsmith, 1992; Konold & Higgins, 2002; Konold, Pollatsek, Well, & Gagnon, 1997; Wilensky, 1997). In the approach that we planned to take, notions such as center, spread-outness, skewness, and relative density would then emerge as ways of characterizing how specific data sets are distributed within this space of values (Bakker & Gravemeijer, 2004; Cobb, 1999; McClain, Cobb, & Gravemeijer, 2000). Furthermore, various statistical graphs or inscriptions would emerge as ways of structuring data distributions in order to identify relevant trends or patterns. Thus, consistent with the third tenet of RME, we viewed the students' development of increasingly sophisticated ways of reasoning about data as inextricably bound up with their development of increasingly sophisticated ways of inscribing data (Biehler, 1993; deLange, van Reeuwijk, Burrill, & Romberg, 1993; Lehrer & Romberg, 1996).

Interviews that we conducted with the 29 students shortly after the design experiment was completed indicate that we had some success in achieving these goals. The interviews included tasks in which

the students were asked to compare graphs of two unequal data sets that corresponded to histograms and to box-and-whiskers plots. A significant majority of the students did so by developing relatively sophisticated arguments that involved reasoning about the data in terms of relative rather than absolute frequencies. In this regard, Konold et al. (1997) argue that a focus on the rate of occurrence (i.e., the relative frequency) of data within a range of values is at the heart of what they term a statistical perspective. As the arguments that most of the students formulated in the interviews involved comparing the graphs in terms of the proportion of data within various ranges of values, it would seem that they were in the process of developing this statistical perspective. It is also worth noting that when we began a follow-up design experiment with some of the same students nine months later, there was no regression in their statistical reasoning (Cobb, McClain, & Gravemeijer, 2003). The students' progress at the beginning of this follow-up experiment was in fact such that they could *all* interpret graphs that corresponded to histograms and to box-and-whiskers plots in these relatively sophisticated ways within the first three or four class sessions of this follow-up experiment.

The retrospective analyses that we have conducted of the design experiment to account for the process of the students' learning and the means by which it was supported highlighted the important role of the instructional activities and the tools the students used to analyze data sets. However, these analyses also indicate the importance of two means of support that are not typically considered by instructional designers. These additional means of support concern the organization of classroom activities and the nature of the classroom discourse (Cobb, 1999; McClain et al., 2000).

Instructional activities

One of the primary commitments that we made when we developed the instructional activities was that students' activity in the classroom should involve the investigative spirit of data analysis from the outset. This implied that the instructional activities should involve analyzing data sets that the students viewed as realistic for purposes that they considered legitimate. As a consequence, most of the

instructional activities that we developed involved comparing two data sets in order to make a decision or judgment (e.g., analyze the T-cell counts of AIDS patients who had enrolled in two different treatment protocols). From the midpoint of the experiment, the students were also required to write reports of their analyses for a specified audience (e.g., the chief medical officer of a hospital in the case of the AIDS data). This requirement reflected the observation that data are typically analyzed with a particular audience in mind almost everywhere except in school (cf. Noss, Pozzi, & Hoyles, 1999).

Tools

The students used two computer tools that were introduced sequentially during the experiment to analyze data sets and thus complete the instructional activities. As these tools have been described extensively elsewhere (e.g., Bakker & Gravemeijer, 2003; Cobb, 2002; McClain, 2002), it suffices to note that they provided the students with a variety of options for organizing graphical inscriptions of data sets. Our intent in designing the tools was that they should fit with the students' reasoning when they were first introduced and that they should support the reorganization of that reasoning as the students used them. Consistent with RME's focus on students' activity, we did not attempt to build the statistical ideas we wanted the students to learn into either the instructional activities or the computer tools in the hope that they might come to see them. Instead, we focused squarely on how the students' use of the tools would change the nature of their activity as they analyzed data and thus the types of statistical reasoning that they might develop.

Organization of classroom activities

In considering the third means of support, the organization of classroom activities, we began to broaden our purview beyond the issues that typically concern instructional designers. One of goals when planning the organization of classroom activities was to ensure that the students would come to view data as measures of an aspect of a phenomenon rather than merely as numbers relatively early in the design experiment. To this end, the teacher introduced each

instructional activity by talking through the data generation process with the students. These conversations often involved protracted discussions during which the teacher and students together framed the particular phenomenon under investigation (e.g., AIDS), clarified its significance (e.g., the importance of developing more effective treatments), delineated relevant aspects of the situation that should be measured (e.g., patients' T-cell counts), and considered how they might be measured (e.g., taking blood samples). The teacher then introduced the data the students were to analyze as being generated by this process. The resulting organization of classroom activities, which often spanned two or more class sessions, therefore involved (a) a whole-class discussion of the data generation process, (b) individual or small-group activity in which the students usually worked at computers to analyze data, and (c) a whole-class discussion of the students' analyses.

In organizing the classroom activities in this manner, we conjectured that as a consequence of participating in the discussions of the data generation process, data sets would come to have a history for the students such that they reflected the interests and purposes for which they were generated (cf. Latour, 1987; Lehrer & Romberg, 1996; Roth, 1997). As it transpired, this conjecture proved to be well founded. On the first instructional activity in which the students used a computer tool to analyze data, approximately half of the students calculated means by hand and selected the data set with the larger mean. However, in the third instructional activity, all the students used the computer tool to identify differences in the data sets that gave insight into the question they were investigating. We interpreted this observation as indicating that doing statistics had come to involve actually analyzing data within the first week of the design experiment (Cobb, 1999; McClain et al., 2000).

In accounting for the effectiveness of the data creation discussions, it is important to note that the teacher did not attempt to teach the students how to generate sound data directly. Instead, she guided the development of a classroom culture in which a premium was placed on the development of data-based arguments. This observation indicates that the concluding whole class discussions in which the students explained and justified their analyses also

played a crucial role. It was as the students participated in these latter discussions that they first became aware of the implications of the data generation process for the conclusions that could legitimately be drawn from data.

Our overriding concern as we prepared for these discussions in the classroom was that mathematically significant issues that advanced the instructional agenda would become explicit topics of conversation. To this end, the teacher and a second member of the research team circulated around the classroom while the students were working at the computers to gain a sense of the various ways in which they were organizing and reasoning about the data. Towards the end of the small-group work, they then conferred briefly to develop conjectures about mathematically significant issues that might emerge as topics of conversation in the subsequent whole-class discussion. Their intent was to capitalize on the students' reasoning by identifying data analyses that, when compared and contrasted, might give rise to substantive mathematical conversations. To the extent that the teacher succeeded, students' participation in the discussions would serve as primary means of supporting their progressive reorganization of their reasoning and thus their gradual induction into the values, beliefs, and ways of knowing of the discipline.

We should clarify that our intent in giving these illustrations from the statistics design experiment has not been to argue that this particular organization of classroom activities should be applied more generally. The crucial contribution of the initial data generation discussions would, for example, appear to be specific to our focus on statistical data analysis. Instead, our purpose has been to demonstrate the importance of attending explicitly to the organization of classroom activities as a primary means of supporting (or inhibiting) students' mathematical learning. In our view, the organization of classroom activities is an integral aspect of an instructional design.

Classroom discourse

In discussing the potential contribution of whole class discussions, we stressed the importance of ensuring that significant mathematical issues that advance the instructional agenda emerge as topics

of conversation. The final means of support that we identified when analyzing the statistics experiment focuses on the nature of classroom discourse. To clarify this means of support, we extend a distinction that Thompson and Thompson (1996) make between calculational and conceptual orientations in mathematics teaching by differentiating between calculational and conceptual discourse. We should stress at the outset that calculational discourse does not refer to conversations that focus on the procedural manipulation of conventional tools and symbols whose use is a rule-following activity for students. The solution methods that students explain as they contribute to calculational discourse might in fact be self-generated and involve relatively sophisticated mathematical understandings. The contrast between calculational and conceptual discourse should therefore not be confused with Skemp's (1976) well-known distinction between instrumental and relational understandings. Instead, the distinction concerns the norms or standards for what counts as an acceptable mathematical argument. In calculational discourse, contributions are acceptable if students describe how they produced a result and they are not obliged to explain why they used a particular method. In contrast to this exclusive focus on methods or solution strategies, the issues that emerge as topics of conversation in conceptual discourse also include the interpretations of instructional activities that underlie those ways of calculating and that constitute their rationale.

As an illustration, the computer tool that the students used to analyze the data on the two AIDS treatment programs provided students with a variety of options for organizing axis plot inscriptions of the patients' T-cell counts (i.e., the T-cell counts for each treatment were inscribed as dots located on an axis of values). The least sophisticated of these options involved dragging a bar to a chosen location on the axis, thereby partitioning the data set into two groups. As shown in Figure 1, the number of points in each group was shown on the screen and adjusted automatically as the bar was dragged along the axis. A calculational explanation of an analysis conducted to determine which treatment program was more effective involves describing the specific steps taken when conducting the analysis. For a relatively unsophisticated analysis in which the bar is used to partition each data set, the students might

simply explain that they placed the bar at a particular value and then report the number of points above and below this value in both data sets. As it so happened, three of 14 groups of students conducted

analyses of this type. In each case, they placed the bar so that what they called the “hill” in one of the data sets was mostly below the bar, and the “hill” in the other data set was mostly above the bar.

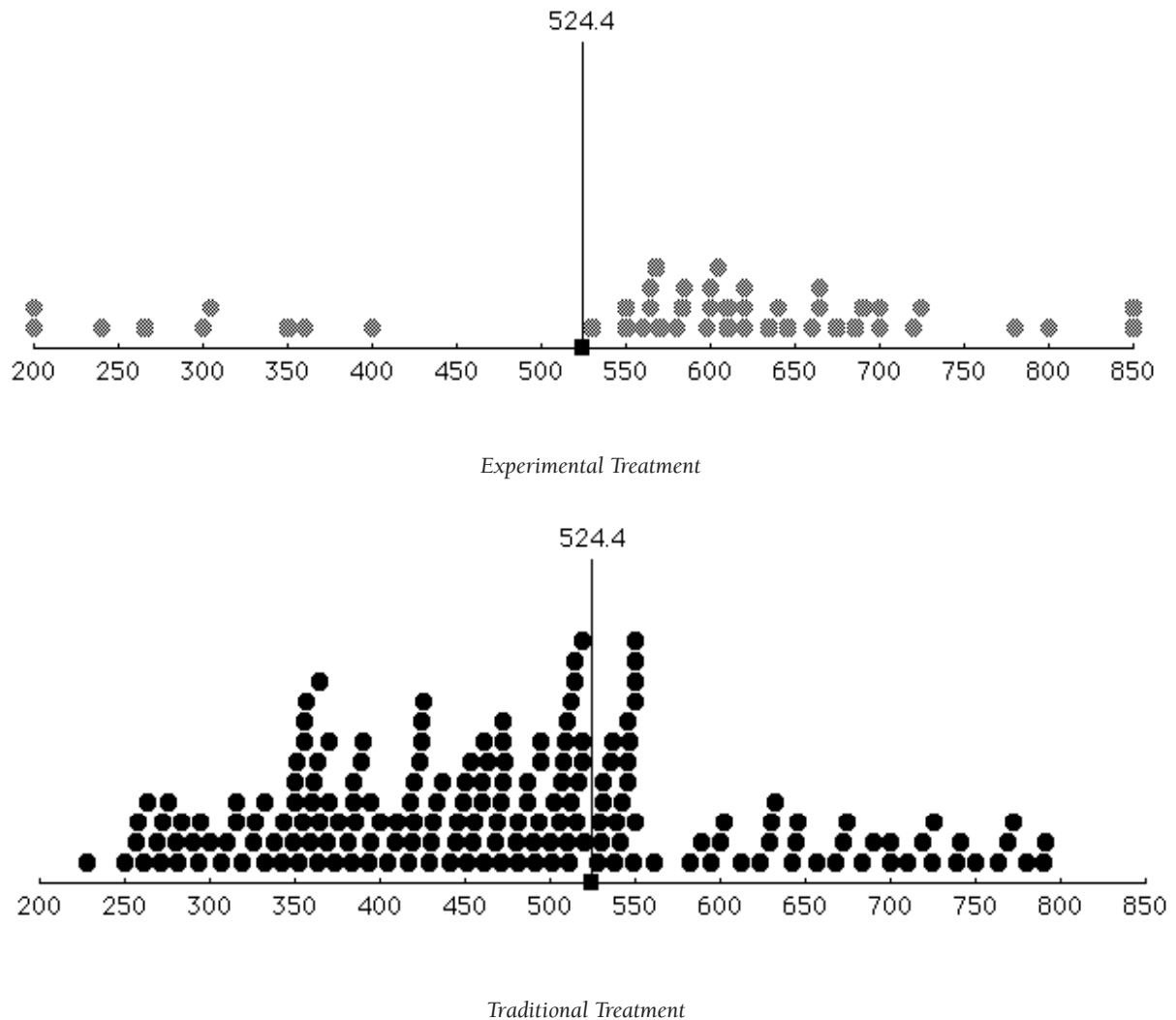


Figure 1. The AIDS Data Partitioned at T-cell counts of 525

A conceptual explanation of these solutions would involve describing not merely the steps of the analysis but also the reasons for carrying them given the issue under investigation, that of judging the effectiveness of the two treatment programs. In giving explanations of this type, the students who participated in the design experiment clarified that they placed the bar at a particular location in order to highlight and quantify a qualitative difference between the two data sets, the location of the “hills.” A retrospective analysis of video-recordings of all classroom sessions indicates that the interventions the teacher made to support conceptual explanations of this type also contributed to the students’ development of relatively sophisticated explanations in which they compared data sets in terms of relative rather than absolute frequencies (Cobb, 1999).

Our experiences in both this and a number of other classroom design experiments lead us to conclude that discussions in which the teacher judiciously supports students’ attempts to articulate their task interpretations can be extremely productive settings for mathematical learning. As these articulations focus on the reasoning that lies behind solution procedures, students’ participation in such discussions increases the likelihood that they might come to understand each other’s reasoning. Had the discussion in the design experiment classroom remained calculational, students could only have understood each other’s explanations by creating a task interpretation that lay behind their use of the computer tool entirely on their own. In contrast, the students’ participation in conceptual discourse provided them with resources that supported their understanding of other’s explanations and thus their reorganization of their initial interpretations of tasks. These resources are not limited to what is said but also include inscriptions and notations that are pointed to and spoken about (cf. A. G. Thompson, Philipp, Thompson, & Boyd, 1994). In the statistics design experiment, for example, the graphs that the students developed as they used the computer tools were integral to communication as well as to their individual reasoning. This interdependency of tools and discourse indicates the systemic nature of the various means of support that we have discussed.

The classroom activity system

As the case of the statistics design experiment indicates, the four means of support that we have

discussed are strongly interrelated. For example, the statistics instructional activities as they were actually realized in the classroom depended on talking through the data generation process, the computer tools that the students used to conduct analyses, and the nature of the subsequent whole class discussions. It is easy to imagine how the instructional activities might be realized differently if the options for organizing data on the computer tool had consisted only of conventional statistical graphs, or if there had been no whole class discussions and the teacher had simply graded the students’ reports of their analyses.

In light of these interdependencies, it is reasonable to view the various means of support as constituting a single classroom activity system. This perspective is compatible with Stigler and Hiebert’s (1999) contention that teaching should be viewed as a system. In making this claim, Stigler and Hiebert directly challenge analyses that decompose teachers’ instructional practices into a number of independent moves or competencies. They instead propose that the meaning and significance of any particular facet of a teacher’s instructional practice becomes apparent only when it is analyzed within the context of the entire practice. In a similar manner, the four means of support that we have discussed should be viewed as aspects of a single classroom activity system (Cobb & McClain, 2004). Instructional design from this point of view therefore involves designing classroom activity systems such that students develop significant mathematical ideas as they participate in them and contribute to their evolution.

This systemic perspective sits uncomfortably with approaches to design that focus exclusively on instructional activities and tools. These formulations focus on selected aspects of the classroom activity system in isolation, in the process casting instructional activities as the cause and learning as the effect. From the systemic perspective, conjectures about what students might learn as they use tools to complete instructional activities are, at best, metonymies for more encompassing conjectures about what students might learn as they participate in an envisioned classroom activity system. This broader perspective brings other aspects of the classroom activity system to the fore as an explicit focus of design, thereby enabling the designer to consider how proposed instructional activities and tools might

be realized in the classroom. In addition, this perspective highlights the central role of the teacher in orchestrating the organization of classroom activities and in guiding the negotiation of norms of mathematical argumentation. A second adaptation that we have made to the RME approach to instructional design emphasizes the crucial contribution of the teacher.

Designing Resources for Classroom Teaching

Instructional designers typically assume that they are developing instructional activities and associated resources to support the learning of individual students. In doing so, they justify instructional sequences in terms of a hypothetical learning trajectory that focuses on the development of individual student's mathematical reasoning. As a consequence of our experience of developing and refining instructional sequences while conducting classroom design experiments, we came to view this exclusive focus on individual students' reasoning as problematic for two reasons. First, our work in classrooms led us to question justifications cast exclusively in terms of individual students' mathematical reasoning for the straightforward reason that, in any classroom, there are significant qualitative differences in students' thinking at any point in time (Cobb et al., 2001). To capture this diversity in students' reasoning, it would be necessary to formulate multiple learning trajectories. However, an approach of this type leads to our second concern about an exclusively individual perspective, namely that it is unmanageable for the teachers. It is unrealistic to expect that teachers will be able to formulate and continually update learning trajectories for every student, or even for several groups of students, and use these multiple trajectories to inform instruction. As an alternative to an exclusively individualistic focus, we have found it useful while working in classrooms to view a hypothetical learning trajectory as consisting of conjectures about the collective mathematical development of the classroom community. This proposal constitutes the second adaptation that we made to the RME design theory.

This adaptation has methodological implications for the analysis of classroom data.² The adaptation is also pragmatically significant and brings the teacher into the picture. In particular, the purpose of instruc-

tional design becomes that of developing resources for the teacher to use to support students' learning, rather than to develop instructional activities that are intended to support students' learning directly. We illustrate this point by returning to the statistics design experiment. We noted that one of the commitments we made when we developed instructional activities during this experiment was that students' activity in the classroom should involve the investigative spirit of data analysis from the outset. A second commitment was that significant mathematical issues that advanced the instructional agenda should emerge as a focus of conversation during the whole-class discussions of the students' analyses. The challenge for us as instructional designers was therefore to transcend what Dewey (1951/1981) termed the dichotomy between process and content by systematically supporting the emergence of key statistical ideas while simultaneously ensuring that the analyses the students conducted involved an investigative orientation. This is a non-trivial issue in that inquiry-based instructional approaches have sometimes been criticized for emphasizing the process of inquiry at the expense of substantive disciplinary ideas. In approaching this challenge, we viewed the various data-based arguments that the students produced as they completed the instructional activities as a primary resource on which the teacher could draw to initiate and guide whole-class discussions that focused on significant statistical ideas. As a consequence, we did not merely attempt to design instructional activities that would be accessible to multiple individual students who may differ in terms of level of statistical sophistication. Our goal when developing specific instructional activities was also to ensure that the diverse ways in which the students analyzed data would constitute an instructional resource on which the teacher could capitalize to support the learning of entire class.

The achievement of this design goal required extremely detailed instructional planning. However, rather than attempting to influence each individual student's reasoning in a specified manner by developing particular instructional activities, we attempted to anticipate the range of data-based arguments that a group of students might produce as they completed specific instructional activities. Our discussion of seemingly inconsequential features of task scenarios and of the particular characteristics of data sets were therefore quite lengthy as minor modifications to an

instructional activity could significantly influence the types of analyses the students would produce and thus the resources on which the teacher could draw to further her instructional agenda. For example, in developing the AIDS instructional activity, we purposefully constructed data sets with a significantly different number of data points so that the teacher could initiate a comparison of analyses that were based on absolute and on relative frequency. In doing so, we paid particular attention to the characteristics of the task scenario in which the inequality in the size of the data sets would seem reasonable to the students. As a result of this detailed preparation, a number of mathematically significant issues emerged during the whole-class discussion of the AIDS data set, including the contrast between absolute and relative frequency, the interpretation of data graphs corresponding to box-and-whiskers plots, and the use of percentages to quantify the proportions of a data set located in particular intervals (Cobb, 1999; McClain et al., 2000).

It is apparent from this illustration that our instructional agenda included a concerted effort to support the students' transition from additive to multiplicative reasoning about data. However, we did not attempt to achieve this goal by somehow causing the individual students to make this transition as they completed certain instructional activities. Instead, we attempted to develop instructional activities that would result in a range of solutions on which the teacher could capitalize as she planned whole class discussions. We therefore viewed ourselves as developing resources that the teacher could use to achieve her instructional agenda. Looking beyond the statistics design experiment, we attribute a central, mediating role to teachers more generally and in fact view them as co-designers of the classroom activity systems that constitute the immediate social situations of their students' mathematical development. It should be clear from the illustrations we have given that the role we envision for the teacher is relatively demanding and requires considerable knowledge and judgment. The third adaptation that we discuss takes these demands seriously and focuses on the possible means of supporting teachers' learning.

Supports for Teachers' Learning

Our characterization of the teachers as co-designers implies that high-quality mathematics teaching

is a generative, knowledge building process in which teachers learn about their students' mathematical reasoning and the means of supporting its development as they analyze, adapt, test, and refine instructional sequences that have proved effective elsewhere (cf. Franke, Carpenter, Levi, & Fennema, 2001). As Gamoran et al. (2003) observe, this form of instructional practice is complex, demanding, uncertain, and not reducible to predictable routines (cf. Ball & Cohen, 1999; Clark, 1988; Lampert, 2001; McClain, 2002; Schifter, 1995; Smith, 1996). Teachers need considerable support as they learn to adapt instructional sequences to the contingencies of their classroom while at the same time placing students' reasoning at the center of their instructional decision-making. The third adaptation that we propose focuses on what is involved when designing instructional sequences so that they support teachers' as well as students' learning.

In our view, teachers' implementation of an instructional sequence is necessarily a process of conjecture-driven adaptation. In taking this stance, we follow de Certeau (1984) in contending that an artifact such as an instructional sequence developed by one group is necessarily reshaped and transformed when others use it. Wertsch (1998) extends this argument by contending that this is the case even when the users attempt to comply with the designers' intentions. To paraphrase Wertsch, teachers necessarily adjust an instructional sequence to the actual circumstances that they encounter even when they attempt to remain faithful to the designers' intentions. Aspects of these circumstances that significantly influence teachers' instructional practices include the institutional constraints that they attempt to satisfy, the formal and informal sources of assistance on which they can draw, their students' prior instructional histories, and the instructional materials and resources that they use as the basis for instruction (Ball & Cohen, 1996; Blumenfeld, Fishman, Krajcik, Marx, & Soloway, 2000; Brown, Stein, & Forman, 1996; Confrey, Bell, & Carrejo, 2001; Feiman-Nemser & Remillard, 1996; Gamoran, Secada, & Marrett, 2000; Nelson, 1999; Rowan, 1990; Senger, 1999; Stein & Brown, 1997; Tharp & Gallimore, 1988).

de Certeau (1984) and Wertsch (1998) take their argument one step further by proposing that

when teachers attempt to use instructional materials developed elsewhere, a second process of creation or production is involved. In our view, this perspective is potentially productive for teacher professional development because it orients us to focus our efforts on supporting teachers' development of the personal and social resources that would enable them to adapt the designed instructional sequences in ways to support their students' learning of significant mathematical ideas. This perspective is also useful for research on teacher professional development in that it orients us to explain why teachers adapt instructional sequences in particular ways by understanding their evolving instructional practices as they are situated in particular institutional settings.

We can best illustrate the third adaptation that we made to RME theory by referring to a teacher professional development experiment³ conducted with a group of middle-school mathematics teachers who taught 12- and 13-year-old students. The statistical instructional sequence that we developed in the prior classroom design experiment served as one of the primary tools that we used to support the teachers' learning. It is important to note that the retrospective analyses that we conducted of the classroom design experiment justify the instructional sequence in terms of (a) a substantiated trajectory for students' mathematical learning, and (b) the documented means of supporting learning along that trajectory. If we had justified the instructional sequence solely with traditional experimental data, the teachers would know that this sequence had proved effective elsewhere but would not have access to the underlying rationale that would enable them to adapt it effectively to their own instructional settings. In contrast, the type of rationale that we developed constitutes a potentially important resource for teachers as they adapt, test, and modify the instructional sequence in their classrooms. Our intent in the professional development experiment was therefore to engage teachers in activities that would enable them to reconstruct the rationale for the instructional sequence.

As an illustration, we engaged the teachers in a variety of activities that focused on how to conduct data generation discussions. As we noted when discussing the statistics design experiment, data generation discussions play a significant role in shaping the ways in which students interpret data.

Early in our collaboration, the teachers recognized that data generation discussions were an important aspect of instruction. However, it became apparent that, from their perspective, effective instructional activities involved a scenario that was immediately interesting and personally relevant to students. For example, the teachers considered that instructional activities that involve soft drinks or roller coasters were instructionally more promising than those that focused on issues of broader social significance (e.g., AIDS). They therefore understood the importance of data generation discussions primarily in terms of capturing students' interest so that they would engage in instructional activities. It was also apparent that, from the teachers' perspective, data generation discussions made little if any contribution to the ways that students interpreted and analyzed data.

During our ensuing collaboration with the teachers, we engaged them in a number of activities that focused on data generation discussions. This approach was reasonably successful in that the teachers came to view a broader range of problem scenarios as potentially productive and saw it as their responsibility to develop the significance and relevance of problem situations with students. Towards the end of our collaboration with the teachers, there were strong indications that the teachers had become aware that the students' understanding of the process by which the data were generated influenced how they interpreted and analyzed the data. In particular, they explicitly linked the issues that they addressed while conducting data creation discussions to students' subsequent analyses. This development would have been unlikely had we not made the rationale for the instructional sequence an explicit focus of professional development activities.

Methodologically, the perspective we have illustrated on teacher learning and implementation offers the prospect that research on teacher professional development might become a design science characterized by tightly integrated cycles in which designs for supporting teachers' learning are developed, tested, analyzed, and modified. The products of teacher development research of this type might include prototypical sequences of activities and resources for teacher-researcher collaboration together with a rationale that is cast in terms of (a) the actual learning trajectory for a professional teaching

community, and (b) means by which that learning can be supported. We speculate that the rationales of this type will enable other researchers and teacher educators to adapt the prototypical sequences to the specific settings in which they are collaborating with teachers in a conjecture-driven manner.

Conclusion

In the first part of this article, we emphasized the value of the positive heuristics for design proposed by RME. These heuristics have not been derived from a general background theory but instead they are empirically grounded in the activity of developing, testing, and revising specific designs in classrooms. The enduring contribution of RME resides in the solution that it proposes to the perennial question of how to induct students into established mathematical practices while simultaneously taking their current understandings and interests seriously. The approach proposed by RME involves continually building up towards substantial participation in established mathematical practices rather than attempting to directly connect students' current understandings to established mathematical ideas and traditions.

Against this background, we discussed three adaptations that we have made to the RME approach to design. The first adaptation involves broadening the means of supporting students' mathematical learning beyond instructional activities and tools to include

the organization of classroom activities and the nature of classroom discourse. These various means of support are strongly interrelated and can be viewed as aspects of an encompassing classroom activity system. The goal for instructional development is therefore to design classroom activity systems such that students develop significant mathematical ideas as they participate in them and contribute to their evolution. The second adaptation emphasized the critical role of teachers as co-designers of classroom activity systems. The purpose in developing instructional activities and tools is not to support individual students' mathematical learning directly. Instead, it is to develop resources that teachers can use to achieve their instructional agendas by capitalizing on students' diverse interpretations and solutions. The third adaptation took account of the demands of the envisioned role for the teacher and concerned supports of teachers' learning. We argued that the implementation of instructional sequences necessarily involves a process of adaptation. The goal for teacher professional development should therefore be to support teachers' development of the personal and social resources that enable teachers to adapt instructional sequences in ways that enhance their students' mathematical learning. This orientation offers the prospect that research on teacher professional development might become a design science that involves developing, testing, and modifying designs for supporting the learning of professional teaching communities and the participating teachers.

ENDNOTES

1. In addition to the first author, the members of the research team for the design experiment were Kay McClain, Koeno Gravemeijer, Maggie McGatha, Jose Cortina, Lynn Hodge, Carrie Tzou, Kazu Nunokawa, Nora Stuart, and Carla Richards.

2. We have argued elsewhere that the appropriate unit of analysis comprises a collective classroom mathematical practice and students' diverse ways of participating in and contributing to its constitution (Cobb et al., 2001).

3. In addition to the authors, the members of the research team for the teacher professional development experiment were Kay McClain, Chrystal Dean, Teruni Lamberg, Lori Tyler, Melissa Gresalfi, and Jose Cortina.

REFERENCES

- Bakker, A., & Gravemeijer, K. (2003). Planning for teaching statistics through problem solving. In R. Charles & H. L. Schoen (Eds.), *Teaching mathematics through problem solving: Grades 6-12* (pp. 105-117). Reston, VA: National Council of Teachers of Mathematics.
- Bakker, A., & Gravemeijer, K. (2004). Learning to reason about distribution. In D. Ben-Zvi & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning, and thinking* (pp. 147-168). Dordrecht, The Netherlands: Kluwer.
- Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal*, 93 (4), 373-397.
- Ball, D. L., & Cohen, D. (1996). Reform by the book: What is - or might be - the role of curriculum materials in teacher learning and instructional reform? *Educational Researcher*, 25 (9), 6-8, 14.
- Ball, D. L., & Cohen, D. (1999). Developing practice, developing practitioners: Towards a practice-based theory of professional education. In G. Sykes & L. Darling-Hammond (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3-32). San Francisco: Jossey-Bass.
- Biehler, R. (1993). Software tools and mathematics education: The case of statistics. In C. Keitel & K. Ruthven (Eds.), *Learning from computers: Mathematics education and technology* (pp. 68-100). Berlin: Springer.
- Blumenfeld, P., Fishman, B. J., Krajcik, J., Marx, R. W., & Soloway, E. (2000). Creating Usable Innovations in Systemic Reform: Scaling Up Technology-Embedded Project-Based Science in Urban Schools. *Educational Psychologist*, 35 (3), 149 - 164.
- Brown, C. A., Stein, M. K., & Forman, E. A. (1996). Assisting teachers and students to reform the mathematics classroom. *Educational Studies in Mathematics*, 31, 63-93.
- Clark, C. M. (1988). Asking the right questions about teacher preparation: Contributions of research on teacher thinking. *Educational Researcher*, 17 (2), 5-12.
- Clement, J., & Brown, D. E. (1989). Overcoming misconceptions via analogical reasoning: Abstract transfer versus explanatory model construction. *Instructional Science*, 18, 327-361.
- Cobb, P. (1999). Individual and collective mathematical learning: The case of statistical data analysis. *Mathematical Thinking and Learning*, 1, 5-44.
- Cobb, P. (2002). Reasoning with tools and inscriptions. *Journal of the Learning Sciences*, 11, 187-216.
- Cobb, P., & McClain, K. (2004). Proposed design principles for the teaching and learning of elementary statistics. In D. Ben-Zvi & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning, and thinking* (pp. 375-396). Dordrecht, The Netherlands: Kluwer.
- Cobb, P., McClain, K., & Gravemeijer, K. (2003). Learning about statistical covariation. *Cognition and Instruction*, 21, 1-78.

- Cobb, P., & Steffe, L. P. (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education*, 14, 83 – 94.
- Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *Journal of the Learning Sciences*, 10, 113-164.
- Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, 31, 175-190.
- Cole, M. (1996). *Cultural psychology*. Cambridge, MA: Belknap Press of Harvard University Press.
- Confrey, J., Bell, K., & Carrejo, D. (2001, April). *Systemic crossfire: What implementation research reveals about urban reform in mathematics*. Paper presented at the annual meeting of the American Education Research Association, Seattle.
- Davydov, V. V. (1988). Problems of developmental teaching (Part I). *Soviet Education*, 30 (8), 6-97.
- Davydov, V. V. (1995). The influence of L. S. Vygotsky on education theory, research, and practice. *Educational Researcher*, 24 (3).
- Davydov, V. V., & Radzikhovskii, L. A. (1985). Vyegtsky's theory and the activity-oriented approach in psychology. In J. V. Wertsch (Ed.), *Culture, communication, and cognition: Vygotskian perspectives* (pp. 35-65). New York: Cambridge University Press.
- de Abreu, G. (2000). Relationships between macro and micro socio-cultural contexts: Implications for the study of interactions in the mathematics classroom. *Educational Studies in Mathematics*, 41, 1-29.
- de Certeau, M. (1984). *The practice of everyday life*. Berkeley: University of California Press.
- deLange, J., van Reeuwijk, M., Burrill, G., & Romberg, T. A. (1993). *Learning and testing mathematics in context. The Case: Data Visualization*. Madison, WI: University of Wisconsin, National Center for Research in Mathematical Sciences Education.
- Dewey, J. (1951/1981). Experience and nature: A reintroduction. In J. A. Boydston (Ed.), *John Dewey: The later works 1925-1953* (Vol. 1, pp. 361-364). Carbondale, IL: Southern Illinois University Press.
- Doise, W., & Mugny, G. (1979). Individual and collective conflicts of centrations in cognitive development. *European Journal of Psychology*, 9, 105-108.
- Dörfler, W. (2000). Means for meaning. In P. Cobb, E. Yackel & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 99-132). Mahwah, NJ: Erlbaum.
- Emirbayer, M., & Mische, A. (1998). What is agency? *American Journal of Sociology*, 103, 962-1023.
- Feiman-Nemser, S., & Remillard, J. (1996). Perspectives on learning to teach. In F. Murray (Ed.), *The Teacher Educator's Handbook* (pp. 63-91). San Francisco: Jossey-Bass.
- Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of teachers' professional development in mathematics. *American Educational Research Journal*, 38, 653-689.
- Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics*, 3, 413-435.
- Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht, The Netherlands: Reidel.
- Gamoran, A., Anderson, C. W., Quiroz, P. A., Secada, W. G., Williams, T., & Ashman, S. (2003). *Transforming teaching in math and science: How schools and districts can support change*. New York: Teachers College Press.
- Gamoran, A., Secada, W. G., & Marrett, C. B. (2000). The organizational context of teaching and learning: Changing theoretical perspectives. In M. T. Hallinan (Ed.), *Handbook of sociology of education* (pp. 37-63). New York: Kluwer Academic/Plenum Publishers.

- Gravemeijer, K. (1990). Context problems and realistic mathematics instruction. In K. Gravemeijer, M. van den Heuvel & L. Streefland (Eds.), *Contexts, free productions, tests, and geometry in realistic mathematics education* (pp. 10-32). Utrecht, The Netherlands: OW & OC Research Group.
- Gravemeijer, K. (1994). *Developing realistic mathematics education*. Utrecht, The Netherlands: CD-β Press.
- Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, 1, 155-177.
- Gravemeijer, K., Cobb, P., Bowers, J. S., & Whitenack, J. W. (2000). Symbolizing, modeling, and instructional design. In P. Cobb, E. Yackel & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 225 – 274). Mahwah, NJ: Erlbaum.
- Hancock, C., Kaput, J. J., & Goldsmith, L. T. (1992). Authentic inquiry with data: Critical barriers to classroom implementation. *Educational Psychologist*, 27, 337-364.
- Holland, D., Skinner, D., Lachicotte, W., Jr., & Cain, C. (1998). *Identity in cultural worlds*. Cambridge, MA: Harvard University Press.
- Konold, C., & Higgins, T. (2002). Working with data: Highlights related to research. In S. J. Russell, D. Schifter & V. Bastable (Eds.), *Developing mathematical ideas: Collecting, representing, and analyzing data*. Parsippany, NJ: Dale Seymour Publications.
- Konold, C., Pollatsek, A., Well, A., & Gagnon, A. (1997). Students' analyzing data: Research of critical barriers. In J. B. Garfield & G. Burrill (Eds.), *Research on the role of technology in teaching and learning statistics: Proceedings of the 1996 International Association for Statistics Education Roundtable Conference* (pp. 151-167). Voorburg, The Netherlands: International Statistics Institute.
- Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University.
- Latour, B. (1987). *Science in action*. Cambridge: Harvard University Press.
- Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. New York: Cambridge University Press.
- Lehrer, R., & Romberg, T. (1996). Exploring children's data modeling. *Cognition and Instruction*, 14, 69-108.
- Lesh, R., & Doerr, H. M. (2000). Symbolizing, communicating, and mathematizing: Key components of models and modeling. In P. Cobb, E. Yackel & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 364-381). Mahwah, NJ: Erlbaum.
- Masingila, J. (1994). Mathematics practice in carpet laying. *Anthropology and Education Quarterly*, 25, 430-462.
- McClain, K. (2002). Teacher's and students' understanding: The role of tools and inscriptions in supporting effective communication. *Journal of the Learning Sciences*, 11, 217-249.
- McClain, K., Cobb, P., & Gravemeijer, K. (2000). Supporting students' ways of reasoning about data. In M. Burke (Ed.), *Learning mathematics for a new century (2001 Yearbook of the National Council of Teachers of Mathematics)* (pp. 174-187). Reston, VA: National Council of Teachers of Mathematics.
- McGatha, M., Cobb, P., & McClain, K. (2002). An analysis of students' initial statistical understandings: Developing a conjectured learning trajectory. *Journal of Mathematical Behavior*, 16, 339-355.
- Nelson, B. C. (1999). *Building new knowledge by thinking: How administrators can learn what they need to know about mathematics education reform*. Cambridge, MA: Educational Development Center.
- Noss, R., Pozzi, S., & Hoyles, C. (1999). Touching epistemologies: Meanings of average and variation in nursing practice. *Educational Studies in Mathematics*, 40 (1), 25-51.
- Nunes, T., Schliemann, A. D., & Carraher, D. W. (1993). *Street mathematics and school mathematics*. Cambridge: Cambridge University Press.
- Pirie, S., & Kieren, T. E. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics*, 26, 61-86.

- Prawat, R. S. (1995). Misreading Dewey: Reform, projects, and the language game. *Educational Researcher*, 24 (7), 13-22.
- Presmeg, N. C. (1997). Reasoning with metaphors and metonymies in mathematics learning. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images*. Hillsdale, NJ: Erlbaum.
- Roth, W. M. (1997). Where is the context in contextual word problems? Mathematical practices and products in grade 8 students' answers to story problems. *Cognition and Instruction*, 14, 487-527.
- Rowan, B. (1990). Commitment and control: Alternative strategies for the organizational design of schools. In C. Cazden (Ed.), *Review of Educational Research* (Vol. 16, pp. 353-389). Washington, DC: American Educational Research.
- Saxe, G. B. (1994). Studying cognitive development in sociocultural context: The development of a practice-based approach. *Mind, Culture, and Activity*, 1, 135-157.
- Schifter, D. (1995). Teachers' changing conceptions of the nature of mathematics: enactment in the classroom. In B. Nelson (Ed.), *Inquiry and the development of teaching: Issues in the transformation of mathematics teaching* (pp. 17-25). Newton, MA: Center for the Development of Teaching, Education Development Center.
- Senger, E. (1999). Reflective reform in mathematics: The recursive nature of teacher change. *Educational Studies in Mathematics*, 37, 199-201.
- Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.
- Sfard, A. (1994, September). *The development of the concept of concept development: From God's eye view to what can be seen with the mind's eye*. Paper presented at the The Symposium on Trends and Perspectives in Mathematics Education, Klagenfurt, Austria.
- Skemp, R. (1976). Relational understanding and instrumental understanding. *Arithmetic Teacher*, 77, 20-26.
- Smith, J. P. (1996). Efficacy and teaching mathematics by telling: A challenge for reform. *Journal for Research in Mathematics Education*, 27, 387-402.
- Steffe, L. P. (1983). The teaching experiment methodology in a constructivist research program. In M. Zweng, T. Green, J. Kilpatrick, H. Pollak & M. Suydam (Eds.), *Proceedings of the Fourth International Congress on Mathematical Education* (pp. 469-471). Boston: Birkhauser.
- Steffe, L. P., & Kieren, T. E. (1994). Radical constructivism and mathematics education. *Journal for Research in Mathematics Education*, 25, 711-733.
- Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-307). Mahwah, NJ: Erlbaum.
- Stein, M. K., & Brown, C. A. (1997). Teacher learning in a social context: Integrating collaborative and institutional processes with the study of teacher change. In E. Fennema & B. Scott Nelson (Eds.), *Mathematics teachers in transition* (pp. 155-192). Mahwah, NJ: Erlbaum.
- Stigler, J. W., & Hiebert, J. (1999). *The teaching gap*. New York: Free Press.
- Streefland, L. (1991). *Fractions in realistic mathematics education. A paradigm of developmental research*. Dordrecht, The Netherlands: Kluwer.
- Tharp, R. G., & Gallimore, R. (1988). *Rousing minds to life*. New York: Cambridge University Press.
- Thompson, A. G., Philipp, R. A., Thompson, P. W., & Boyd, B. A. (1994). Computational and conceptual orientations in teaching mathematics. In *1994 Yearbook of the National Council of Teachers of Mathematics* (pp. 79-92). Reston, VA: National Council of Teachers of Mathematics.
- Thompson, A. G., & Thompson, P. W. (1996). Talking about rates conceptually, part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education*, 27, 2-24.
- Thompson, P. W. (1992). Notations, principles, and constraints: Contributions to the effective use of concrete manipulatives in elementary mathematics. *Journal for Research in Mathematics Education*, 23, 123-147.