

## APPENDIX II<sup>1</sup>

### Free Logics

The aim of this appendix is to present a short overview of free logics from the modeltheoretical point of view. For alternative non model-theoretical approaches see Rahman/Rückert/Fischmann (1997), Rahman (2001) and Fontaine/Redmond/Rahman (2009).

One of the first formal developments towards the logic of non-existents is the one of free logics. Free logic is shorthand for logic free of *existence assumptions*. There are two existence assumptions built into classical logic, namely

- The domain of quantifiers is not empty. Thus, the following holds:  
 $\exists x(Ax \vee \neg Ax)$   
 $\exists x(x=k)$ .
- Every term denotes. This assumption renders the following valid (in classical logic)  
Infer  $\phi[k]$  from  $\forall x\phi$   
Infer  $\exists x\phi$  from  $\phi[k]$   
Infer  $\exists y \neg \exists x(x=y)$  from  $\neg \exists x(x=k)$ .

Not every free logic rejects the first existence assumption. Those that do are known as *universally free logics*. Existential generalization and universal specification hold under the restriction that the terms involved exist. Usually, the assumption of existence is made explicit by the means of a first-order existence-predicate (E!).

Karel Lambert [(1960), who penned the expression *free logic*, distinguishes three types of free logics, negative, positive and neuter. Ermanno Bencivenga (1986) added a new type based on supevaluations. Let us start with the description of the first three.

#### AII.1. Negative, positive and neuter free logics

**Negative free logic**, allows constants not refer at all. The identity axiom holds under the same restriction as existential generalisation and universal instantiation. Atomic formulae in which constant occur that do not refer are false.

Formulae  $\phi$  are evaluated in models  $M = (D, I)$  relative to variable assignments  $\gamma$  and the interpretation function  $I$ , that, when applied to constant it is a *partial* function, i.e. to each constant  $k$  and to some members of  $D$ ,  $I$  assigns a member of  $D$ . In other words,  $I(k)$  may not be defined, but if it is, then  $I(k) \in D$ . While  $I$ , when applied to constant is a partial function, assignments are total functions. In other words, if  $x$  is free variable the value of  $\gamma(x)$  is an element of  $D$ .

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<sup>1</sup> Part of the content of this appendix has been extracted from : Fontaine/Redmond/Rahman 2009, though the semantics have been adapted to the objectual interpretation of quantifiers.

The relevant truth conditions are

- (i)  $V_M(Pk_1, \dots, k_n) = 1$  iff.  $I(k_1), \dots, I(k_n)$  are defined and  $\langle I(k_1), \dots, I(k_n) \rangle \in I(P)$ .
- (ii)  $V_M(k_i = k_j) = 1$  iff.  $I(k_i)$  et  $I(k_j)$  are defined and  $I(k_i)$  yields the same element that  $I(k_j)$ .
- (iii)  $V_M(E!k_i) = 1$  iff  $I(k_i)$  is defined.

The rest of the truth-conditions are standard.

Thus, if one the function  $I$  is undefined for at least one  $k_i$ , the following are all false:

$$\begin{aligned} k_1 &= k_1 \\ k_1 &= k_2 \\ Pk_1 \end{aligned}$$

**Neuter free logic**, is a negative free logic that allows formulae to be neither false nor true. That is if a formula contains a constant that does not refer then the whole formula is neither true nor false. While positive and negative free logic do not induce changes to classical propositional logic (without equality), neuter free logic does:  $Ak \vee \neg Ak$  is not generally valid. That is, interpretations are also defined as partial functions though, different to the case of negative free logic; a valuation might yield truth-value gaps:

The relevant truth conditions are

- (i)  $V_M(Pk_1, \dots, k_n) = 1$  if.  $I(k_1), \dots, I(k_n)$  are defined and  $\langle I(k_1), \dots, I(k_n) \rangle \in I(P)$   
otherwise  $V_M(Pk_1, \dots, k_n)$  is undefined
- (ii)  $V_M(k_i = k_j) = 1$  if  $I(k_i)$  et  $I(k_j)$  are defined and  $I(k_i)$  yields the same element that  $I(k_j)$   
otherwise  $V_M(k_i = k_j)$  is undefined.
- (iii)  $V_M(E!k_i) = 1$  iff  $I(k_i)$  is defined.

**Positive free logic**, allows singular terms to refer to non-real objects. The domain might contain real and non real elements. The result is that the identity axiom holds in any such logic extended with equality. That is, there might be identity of non-existent objects. Furthermore in positive free logic we might introduce two pairs of quantifiers: ontologically committed quantifiers and ontologically not committed quantifiers.

The most important semantics for positive logic partitions the domain  $D$  in two, namely an *outer* domain and an *inner* domain. The inner domain is as in classical logic (except that we may allow it to be empty – in the case the logic ought to be universally free): it contains real existing objects. The outer domain consists of the references of terms designating non-existents such as Pegasus, King Lear, Martin Fierro. Every term refers (either in the outer or in the inner domain). More precisely a model is a triple  $\langle D_I, D_O, I \rangle$  where  $D_I$  is the internal domain,  $D_O$  the external domain and  $I$  is the interpretation function defined over the whole domain, that is over the union of inner with the outer domain. The ontological-charged quantifiers range only over the inner domain  $D_I$ . A second pair of quantifiers, non-ontologically charged, can also be defined: their range is  $D_I \cup D_O$ .

*Thus, I is defined as follows:*

- (i) For any constant  $k$   $I(k)$   $k$  is an element of  $D = D_I \cup D_O$ .
  - (ii) For any predicate of  $n$ -places  $P$ , **including identity**,  $I(P)$  is the set of  $n$ -tuples of members of  $D = D_I \cup D_O$ .
- An assignment  $\gamma$  in a constant (varying) domain first-order model  $M$  is a mapping that assigns to each free variable  $x$  some member  $\gamma(x)$  of  $D = D_I \cup D_O$
  - Let  $\gamma$  and  $\gamma'$  be two assignments. We say that  $\gamma'$  is an  **$x^*$ -variant** of  $\gamma$  if both assignments agree on all variables except possibly the variable  $x$  **and**  $\gamma'(x)$  is a member of  $D_I$ .

The point is that free variables that can find an assignment somewhere in  $D$  (exactly like interpretations) but the ontologically charged quantifiers will be defined with the help of a special kind of *variants*, restricted to the inner domain, in such a way that the quantifiers only range over  $D_I$ .

Thus, we have:

- (i)  $V_M, \gamma (\forall x \phi) = 1$  iff for every  **$x^*$ -variant**  $\gamma'$  of  $\gamma$   $V_M, \gamma' (\phi[x/\delta]) = 1$ , where  $\delta \in D_I$ .
- (ii)  $V_M, \gamma (\exists x \phi) = 1$  iff for some  **$x^*$ -variant**  $\gamma'$  of  $\gamma$   $V_M, \gamma' (\phi[x/\delta]) = 1$ , where  $\delta \in D_I$

The introduction of a second pair of non-ontologically charged quantifiers is straightforward, we need only to define assignment variants that range over  $D = D_I \cup D_O$  and not only over  $D_I$

In appendix AII.3 we will describe a semantics of positive logic with inner and outer domains in the context of first order modal logic.

## **AII.2. Supervvaluations and Superinterpretations**

### **AII.2.1. Supervvaluations**

From the point of fiction irrealists claim that names of fictions are empty, they do not refer at all. In this sense, they have the feeling that that positive free logic cheats. When irrealists say a name is empty, that it refers to nothing, they do not mean that it refers, but to something which does not exist; they mean that it does not refer at all. However, the gap values of the neuter free logic might make it difficult to define a notion of logical consequence involved. The classical account of consequence says that one proposition is a consequence of others provided no interpretation leads from the truth of the latter to the falsity of the former. But this definition will allow inferences that are rejected in free logics. Indeed, take existential generalization: If  $\phi[k]$  is true then  $\exists x \phi$  must be true, if  $\phi[k]$  lacks a truth-value (i.e. when  $k$  is empty),  $\exists x \phi$  lacks it too. The inference did not lead from truth to falsity. Thus, according to this notion of consequence, existential generalization is valid. This is not right. To excluded unwanted inferences such as the one mentioned above, we need to rule out the move from lack of value to falsity (or lack of value) as invalid. That is, it seems that we ought to say that one proposition is a consequence of others if no interpretation leads from propositions none of which are false to one which is false or lacks a value. Unfortunately, as pointed out by

Stephen Read (1995), this revised criterion invalidates inferences which we wish to class as valid.<sup>2</sup> Consider a model where  $Bk_2$  is false (i.e.  $k_2$  refers but it does not satisfy B) and  $Ak_1 \wedge \neg Ak_1$  lacks a value (since  $k_1$  does not refer). Thus the inference from  $Ak_1 \wedge \neg Ak_1$  to  $Bk_2$  leads from lack of value to falsity, and so will be invalid.<sup>3</sup>

One way out, for irrealists, is to make use of Bas van Frassens' (1966) method of supervaluations. Moreover, this approach seems to fit very well with the make-believe theory on fiction.

According to the theory of supervaluations models consist of *partial valuations*, *classical extensions* and *supervaluations*.

- Partial valuations allow truth-value gaps such as in some three-valued logics In other words an assignment of truth to some propositions, falsity to others and not value to the rest.
- Consider all ways of extending this partial valuation to a total valuation by arbitrarily assigning values (consistent with the truth-conditions - if a given proposition is arbitrarily made true, then any disjunction containing this proposition will be made true too) to those propositions that the partial function yields a lack of value. Call these, the *classical extensions* of the original partial valuation.
- A *supervaluation* is defined as follows :  
A proposition is true according to the supervaluation if it is true in all classical extensions, false according to the supervaluation ; if it is false in all classical extensions and has no (super-)value if it takes different values in different classical extensions.

**Logical consequence:** A proposition is a logical consequence of other propositions if there is no partial valuation every classical extension of which makes all the premises true and the conclusion false

**Validity:** A proposition is valid according to supervaluation if there is no partial valuation the classical extension of which renders that proposition false.

Take some instance of the principle of non-contradiction where we assume that some empty name occurs, and a partial valuation as described in the matrix below by the lines 1 to 4.

1	$\phi$	$\neg\phi$	$\neg(\phi \wedge \neg\phi)$
2	1	0	1
3	#	#	#
4	0	1	1
<b>5</b>	<b>1</b>	<b>0</b>	<b>1</b>
<b>6</b>	<b>0</b>	<b>1</b>	<b>1</b>

The first of the two possible classical extensions (line 5) assign true to  $\phi$  and accordingly false to its negation while the second extension (line 6) assigns the dual values. According to

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<sup>2</sup> Read 1995, 139.

<sup>3</sup> Read 1995, 139.

both of these extensions  $\neg(\phi \wedge \neg\phi)$  is true, and thus so is its supervalue. Non-contradiction is thus valid according to supervaluation.

One way to read supervaluation is to read classical extension as *if-valuations*. In other words, some propositions are neither true nor false, e.g. those involving fictional terms, but we do- by means of classical extensions, as if they were true or false. This reading might provide the semantics of the make-believe approach. However, the framework is still incomplete. Supervaluations are only efficient at the propositional level. What about quantifiers? What about  $k_i = k_i$ . Is it true ; false or lacks of a value? If we consider equalities to be atomic propositions then identity will lack a supervalue. Bencivenga accomplished the task to extend the supervaluational framework to first order free logic by combining outer domains with supervaluations.

### ***AII.2.2. Superinterpretation: Combining outer domains with supervaluations***

**Superinterpretation** : Bencivenga (1986) does not consider all classical extensions. Instead he considers all ways of assigning a denotation to the empty terms, and the total valuations which will result from that. However, the denotations of the empty terms are chosen not from the domain of the partial valuation, but from arbitrary extensions of that domain. Woodruff (1971, 1984) developed the view that extensions of denotations should be thought as extended interpretations that find their values in an added outer domain. We will follow Woodruff's account. .

- A *free extension* of a partial valuation comprises an extension of the domain by the addition of (non-empty) outer domain, together with an extension of the interpretation of the predicate letters to the outer domain and the resulting total valuation resulting from these extensions.<sup>4</sup>
- Supervaluation, consequence and validity are defined as before but by substituting *classical extension* with *free extension*.

It is important to point out that the approach of Bencivenga still yields a free logic and not a classical logic. Let us see the details for the rejection universal specification within this framework:

Consider  $U'$  to be the result of a free extension of model  $U$  in relation to the constant  $k$ . In  $U$  we might have :

$$V_U(\forall xPx) = 1 \text{ and, if we assume that } k \text{ is an empty name : } V_U(Pk_1) = \#$$

In  $U'$  we will have free extensions such as

$$V_{U'}(\forall xPx) = 0 \text{ (or } V_{U'}(\forall xPx) = 1) \text{ et } V_{U'}(Pk_1) = 0 \text{ (or } V_{U'}(Pk_1) = 1) - \text{ recall that in } U' \text{ while the interpretation of the constants have values in the outer domain, quantifiers range over the inner domain.}$$

What values should we then chose, those of  $U'$  or of  $U$ ? The idea is that, if the  $U$ -valuation does not yield lack of value, then that  $U$ -valuations should be given priority. Thus, in our case

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<sup>4</sup> Cf. Read 1995, 138-139.

we will retain  $V_U(\forall xPx) = 1$ , but switch to  $U'$  for the consequent. Now, for the consequent we have as one of the possible free extensions  $V_{U'}(Pk_1) = 0$ . Hence, universal specification fails to hold.<sup>5</sup>

In other words, the semantics of superinterpretation starts with an interpretation function defined as in neuter free logic but later on, the truth-value gaps produced by the partial function are filled up in such a way that the result is a positive logic.

More generally, according to superinterpretations, we do as if empty names denote some non-existent. Empty names are empty and the correspondent propositions might lack truth-value but if we pretend that they refer some non-existent object of a pretended outer domain, then we might consider those propositions to be true.

### AII.3. Free logics in a modal context

The aim of this appendix is to develop a semantics of first order modal logic able to display a positive free logic without empty domains with and without constant domains.<sup>6</sup> The varying domain frames, could be seen as furnishing the basis of a modal Meinongianism framed on normal worlds. The extension to non-normal worlds should be straightforward. The advantage of varying domains is that it might be seen as implementing a kind of “creationism” within Meinongianism. Now, on our view, this will not be fully accomplished until the created elements of each world are taken seriously as creations in the sense Thomasson (see last part of the paper above).

#### Varying and constant domains

Let us have the frame  $\langle W, R, D \rangle$  for first-order modal logic

The domain could be constant or varying

#### Constant Domains:

- We say that the domain of the frame  $\langle W, R, D \rangle$  is *globally constant* if  $D$  is a non-empty set, called the domain of the frame over which quantifiers can range, no matter at what world

But we could also have the following weaker version:

- We say that the domain of the frame  $\langle W, R, D \rangle$  is *locally constant* if  $D$  is a non-empty set, such that for for  $w, w' \in W$  such that  $wRw'$ , then  $D(w) = D(w')$ .

Barcan and converse Barcan-formulae are valid in locally constant domain models. Thus the simultaneous validity of the Barcan- and converse-Barcan formulae characterize locally constant domain models.

More generally, a sentence is valid in all locally constant domain models iff this sentence is valid in all globally constant models

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<sup>5</sup> Cf. Woodruff 1971.

<sup>6</sup> Cf. Fitting/Mendelson 1998 and Garson 2006

### Varying Domains:

- We say that the domain of the frame  $\langle W, R, D \rangle$  is *globally varying* if  $D$  is a non-empty set, such that for any  $w, w' \in W$ , then it is not always the case that  $D(w) = D(w')$ .

In such domains neither the Barcan ( $\Diamond \exists x A x \rightarrow \exists x \Diamond A x$ )- nor the converse Barcan ( $\exists x \Diamond A x \rightarrow \Diamond \exists x A x$ )- formulae are valid.

Here too some weaker versions are available:

- We say that the domain of the frame  $\langle W, R, D \rangle$  is *monotonic (or increasing)* if  $D$  is a non-empty set, such that for any  $w, w' \in W$ , such that  $w R w'$ , then  $D(w) \subseteq D(w')$ .

The converse Barcan-formulae, such as  $\exists x \Diamond A x \rightarrow \Diamond \exists x A x$ , characterize such a domain.

- We say that the domain of the frame  $\langle W, R, D \rangle$  is *anti-monotonic (or decreasing)* if  $D$  is a non-empty set, such that for any  $w, w' \in W$ , such that  $w R w'$ , then  $D(w') \subseteq D(w)$   
(Note that the order of inclusion has been reversed)

The Barcan-formulae, such as  $\Diamond \exists x A x \rightarrow \exists x \Diamond A x$ , characterize such a domain

For varying domains we have one first important decision to take: should we allow to have singular terms and free variables in our language that may or may not be in the domain of some possible world? If we do not allow this to happen then varying domains become very difficult to handle and quite a big amount of ad hoc clauses have to be implemented (See Gamut vol. II). We will take rather the second choice and allow to have singular terms in our language that may or may not be in the domain of some possible world. In this case two main approaches are possible:

- the positive free logic one (terms need not to designate in every world of the frame but in at least one world ) and
- the deflationists (neuter and negative free logics): things which do not exist can not be referred to or mentioned, no statement can be about them

The argument in favour of the positive free logic approach comes from the fact that if we allow to have singular terms in our language that may or may not be in the domain of some possible world, what should we say of the truth of a formula  $Pk$  in world  $w$  where the value assigned to  $k$  is not an element of the domain of  $w$ ? In such a set up we have three reasonable choices

- 1) allways take  $Pk$  to be false at  $w$ .
- 2) leave the truth of  $Pk$  undetermined
- 3) make no special restrictions: in particular  $Pk$  could be true though  $k$  does not exist in the world at stake:

The first option is only sensible if it applies to atomic formulae: notice that we do not want to say that the negation of  $Pk$  is false if  $k$  does not exist at  $w$  (because  $Pk$  is false too). But

Kripke (1963, p. 85, footnote 1) has observed that if we take solution 1 imposing this requirement on atomic sentences leads to a modal logic without uniform substitution. The real options are positions two or three. We will follow position 3. It amounts to the rejection of the classically valid formulae:

$$\forall xAx \rightarrow Ak$$

$$Ak \rightarrow \exists xAx$$

The point is that the value assigned to  $k$  might lie beyond the values assigned to the variables of the quantifiers at the world at stake, say  $w$ : the value assigned to  $k$  might be an element of the domain of  $w'$  and not of  $w$ .

**Inner and outer domains in the modal setting:**

In other words: the idea of outer domains is implemented as distinguishing between the interpretations that yields values in the domain of the world at stake, say  $w$ , (the domain of  $w$  corresponds then to the inner domain) or values in the domain of a different world  $v$  (the domain of  $v$  corresponds to the outer domain of  $w$ ).

**Definition 1: Domain of the frame:**

The domain of the frame  $D_F$  is the union of the domains of all the possible worlds of the domain

**Definition 2: Interpretation in constant and varying domains:**

The interpretation  $i$  is an *interpretation* in a constant (varying) domain frame  $F: \langle W, R, D \rangle$  if  $i$  assigns, to each  $n$ -place relation symbol  $R$  of the language, and to each possible world  $w$  of  $W$  some  $n$ -place relation on the domain of the frame  $D_F$ .

**Definition 3: Model in constant and varying domains:**

A constant (varying) domain first-order model is a structure  $M = \langle W, R, D, i \rangle$  where  $\langle W, R, D \rangle$  is the correspondent first-order frame and  $i$  the appropriate interpretation

**Definition 4: Assignment in constant and varying domains:**

An assignment  $\gamma$  in a constant (varying) domain first-order model  $M$  is a mapping that assigns to each free variable  $x$  some member  $\gamma(x)$  of  $D_F$ .

**Definition 5: Variant in constant domains:**

Let  $\gamma$  and  $\gamma'$  be two assignments. We say that  $\gamma'$  is an  $x$ -variant of  $\gamma$  in constant domain if both assignments agree on all variables (including  $k$ -terms) except possibly the variable  $x$ .

**Definition 6: Variant in varying domains:**

Let  $\gamma$  and  $\gamma'$  be two assignments. We say that  $\gamma'$  is an  $x^*$ -variant of  $\gamma$  if both assignments agree on all variables (including  $k$ -terms) except possibly the variable  $x$  (where  $x$  is not a  $k$ -term) and  $\gamma'(x)$  is a member of  $D^w$



- The point is that in varying domain a *\*variant* at a world  $w$  can not assign objects that are beyond the domain of  $w$ , if quantifiers should range only over the domain of the world at stake. Furthermore, notice that this restriction does not apply free variables that can find an assignment somewhere in the domain of the frame.

**Definition 7: Truth in a model with varying domains**

The point that the variants defining the truth in the model for quantifiers in varying domains will assign objects of the domain of the world at stake. The relevant definitions are those of positive logic (see appendix AII.2.1).

Thus, we have:

- (i)  $V_{M, w, \gamma} (\forall x \phi) = 1$  iff for every ***x\*-variant***  $\gamma'$  of  $\gamma$   $V_{M, w, \gamma'} (\phi[x/\delta]) = 1$ , where  $\delta \in D^w$
- (ii)  $V_{M, w, \gamma} (\exists x \phi) = 1$  iff for some ***x\*-variant***  $\gamma'$  of  $\gamma$   $V_{M, w, \gamma'} (\phi[x/\delta]) = 1$ , where  $\delta \in D^w$

**Final remark:** We have chosen to describe positive free logic for varying domains. In fact, to implement positive free logic in a constant domain setting is straightforward: it is sufficient to introduce a predicate of existence and distinguish with the help of this predicate the inner and outer domain of each world.