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ANTAGONISTIC GAMES

Jean-Pierre BEAUD *

RÉSUMÉ – Jeux antagonistes

Certaines propriétés des jeux à somme nulle sont étendues à différentes classes de jeux à deux joueurs à somme non nulle. Les définitions de ces classes sont basées sur différentes notions d'antagonisme. La notion de jeux presque strictement compétitifs, introduite par Aumann en 1961, est généralisée en gardant l'unicité du paiement d'équilibre de Nash. L'ensemble des équilibres de Nash d'un jeu faiblement unilatéralement compétitif coïncide avec l'ensemble de ses points selles. Quelques propriétés sont vérifiées pour les paiements d'équilibre corrélé d'un jeu faiblement unilatéralement compétitif; et pour les équilibres de Nash d'un jeu symétrique faiblement unilatéralement compétitif.

MOTS CLÉS – Antagonisme, Jeu à somme nulle, Unicité du paiement d'équilibre.

SUMMARY – *Some properties of zero-sum games are extended to different classes of two-person non zero-sum games. The definitions of these classes are based on different notions of antagonism. The notion of almost strictly competitive games introduced by Aumann in 1961, is generalized while keeping the uniqueness of Nash equilibrium payoff. The set of Nash equilibria of a weakly unilaterally competitive game coincides with the set of its saddle points. Some properties are satisfied by the correlated equilibrium payoff of a weakly unilaterally competitive game; and by Nash equilibria of a symmetric weakly unilaterally competitive games.*

KEYWORDS – Antagonism, Two-person zero-sum games, Uniqueness of equilibrium payoff.

1 INTRODUCTION

When Nash equilibria of a zero-sum game exist, they satisfy interesting properties: they induce the same payoff, they are interchangeable, the set of Nash equilibria is convex, even the correlated equilibrium payoff is unique. Generally, these properties do not extend to non zero-sum games. Nevertheless, some classes of two-person games satisfying part of these properties have been introduced by different authors [1, 8, 9]. These classes are defined by extending the conflicting aspect of the zero-sum case: for the classes introduced by Friedman (strictly competitive game) and Kats and Thisse

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((weakly) unilaterally competitive games), competition is defined by comparing couples of strategies according to comparison rules based on payoffs. The class introduced by Aumann (almost strictly competitive games [1]) is defined through the notion of twisted equilibrium with the aim of comparing the strategic behavior of the players.

Section 2 is devoted to notations. In section 3, after recalling some properties of two-person zero-sum games, the classes introduced by Aumann, Friedman, Kats and Thisse are defined. We generalize Aumann's class of games. In section 4, we show that all these games have a unique Nash equilibrium payoff. In section 5, we are interested in the "correlated equilibrium" introduced by Aumann in 1974 [2]. Every correlated equilibrium of a zero-sum game was proved to induce the same payoff (see [16]). We extend this property to weakly unilaterally competitive games. In section 6, we stress the difference between antagonism in pure and in mixed strategies. Repeated games are the subject of section 7. Symmetric games is the subject of section 8. We give a few examples in section 9.

2 NOTATIONS

Let $\mathbb{N}$ denote the set of integer numbers and $\mathbb{R}$ the set of real numbers. For any finite set L , Δ^L is the set of probability distributions on L and $\#L$ stands for its cardinality.

A two-person game denoted $G = (S_1, S_2, u_1, u_2)$ consists of a set of strategies S_i for each player i and a payoff function u_i ; $u_i : S_1 \times S_2 \rightarrow \mathbb{R}$. Let $-i$ be the player $3 - i$ (*i.e.* $-i$ is the opponent of player i) and N be the set of players; $N = \{1, 2\}$.

The set of Nash equilibria of the game G is denoted by $NE(G)$, and $NEP(G)$ is the set of its Nash equilibrium payoffs.

HYPOTHESIS 1. *The sets of strategies S_i and the payoff functions u_i are such that the game (S_1, S_2, u_1, u_2) has one Nash equilibrium.*

Let $v = (v_1, v_2)$ be the *threat point*, with

$$v_i = \inf_{S_{-i}} \sup_{S_i} u_i(s_i, s_{-i}) \quad (1)$$

A game G is zero-sum if $u_2 = -u_1$. We simply note $G = (S_1, S_2, u_1)$. When $\inf_{S_2} \sup_{S_1} u_1(s_1, s_2) = \sup_{S_1} \inf_{S_2} u_1(s_1, s_2)$, it is the value of G and denoted $\text{Val}u_1$. We define similarly $\text{Val}u_2$.

3 SOME CLASSES OF TWO-PERSON GAMES

3.1 ZERO-SUM GAME

PROPERTY 3.1 *Let $G = (S_1, S_2, u)$ be a zero-sum game. Then:*

1. $NEP(G) = \{v\}$;

2. $NE(G)$ is convex if S_1 and S_2 are convex and for every $(s_1, s_2) \in S_1 \times S_2$, u is quasi-concave in s_1 and quasi-convex in s_2 ;
3. The equilibria are interchangeable, i.e. if (s_1, s_2) and (s'_1, s'_2) are equilibria, so are (s_1, s'_2) and (s'_1, s_2) ;
4. The correlated equilibrium payoff is unique.

These properties are no longer true for non zero-sum games. This leads us to define classes of non zero-sum games possessing part of these properties. They are defined in two ways:

1. Two couples of strategies are compared according to different comparison rules (games of type A [8], B and C [9])
2. We compare the strategic behavior of the players through the concept of twisted equilibrium (see Section 3.5).

3.2 GAME OF TYPE A (OR STRICTLY COMPETITIVE)

The class of strictly competitive games was the first class of non zero-sum games introduced to generalize some results of zero-sum games ([14], [8]).

DEFINITION 3.1 G is a game of type A if for all $s, s' \in S_1 \times S_2$,

$$u_1(s) \geq u_1(s') \Leftrightarrow u_2(s) \leq u_2(s') \quad (2)$$

Any zero-sum game is of type A . The converse is not true:

EXAMPLE 3.1 $G_1 = (S_1, S_2, u_1, u_2)$ with $u_2 = -2u_1$.

We will see that if G is a game of type A , then $\alpha u_1 + \beta u_2 = \text{constant}$ for some positive real numbers α, β .

3.3 GAME OF TYPE B (OR UNILATERALLY COMPETITIVE)

This class of games was introduced by Kats and Thisse in 1992 under the name “unilaterally competitive game” [9].

DEFINITION 3.2 (Kats-Thisse, 1992) G is a game of type B if for all $i \in N$, for all $s_i, s'_i \in S_i$, $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \Leftrightarrow u_{-i}(s_i, s_{-i}) \leq u_{-i}(s'_i, s_{-i}) \quad (3)$$

For games of type B , we only consider unilateral deviation, so every game of type A is of type B . The converse is not true:

EXAMPLE 3.2 *Prisoner's dilemma*

$$G_2 = \begin{array}{cc} & \begin{array}{cc} A_2 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ B_1 \end{array} & \begin{pmatrix} 1, 1 & 4, 0 \\ 0, 4 & 3, 3 \end{pmatrix} \end{array}$$

3.4 GAME OF TYPE C (OR WEAKLY UNILATERALLY COMPETITIVE)

This class was defined by Kats and Thisse [9] by weakening (3) in the following sense:

DEFINITION 3.3 (Kats-Thisse, 1992) *G is a game of type C if for all $i \in N$, for all $s_i, s'_i \in S_i, s_{-i} \in S_{-i}$,*

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i}) \Rightarrow u_{-i}(s_i, s_{-i}) \leq u_{-i}(s'_i, s_{-i}) \quad (4)$$

and

$$u_i(s_i, s_{-i}) = u_i(s'_i, s_{-i}) \Rightarrow u_{-i}(s_i, s_{-i}) = u_{-i}(s'_i, s_{-i}) \quad (5)$$

If an unilateral change of strategy of player i leads to a strict gain for him, then it induces a weak loss for player $-i$; and if player i is indifferent, so is player $-i$.

Every game of type B is of type C. The converse is not true:

EXAMPLE 3.3

$$G_3 = \begin{array}{cc} & \begin{array}{cc} A_2 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ B_1 \end{array} & \begin{pmatrix} 1, 1 & 1, 0 \\ 0, 1 & 0, 0 \end{pmatrix} \end{array}$$

Player 1 strictly prefers (A_1, B_2) to (B_1, B_2) whereas player 2 is indifferent between these two couples of strategies.

These classes are defined directly by the data of the game (*i.e.* the sets of strategies and the payoff functions) with no need of another concept. This is no longer the case for the classes we introduce now .

3.5 GAME OF TYPE I (OR ALMOST STRICTLY COMPETITIVE)

This class of games was introduced by Aumann in 1961 under the name “almost strictly competitive game” [1]. Its definition is based on the notion of twisted equilibrium.

DEFINITION 3.4 (Aumann, 1961) $\bar{s}$ is a twisted equilibrium of $G = (S_1, S_2, u_1, u_2)$ if $\bar{s}$ is a Nash equilibrium of $\bar{G} = (S_1, S_2, -u_2, -u_1)$.

$\bar{G}$ is called the twisted game. Player i has the same set of strategies in G and $\bar{G}$ but his payoff in $\bar{G}$ is the opposite of the payoff of player $-i$ in G . We denote $TE(G)$ the set of twisted equilibria of G and by $TEP(G)$ the set of twisted equilibrium payoffs *i.e.* $TEP(G) = \{(t_1, t_2) \in \mathbb{R}^2 : \exists \bar{s} \in TE(G), u_i(\bar{s}) = t_i\}$. $TEP(G)$ is the set of payoffs in G of equilibrium of $\bar{G}$.

DEFINITION 3.5 (Aumann, 1961) *G is a game of type I if G satisfies:*

- (i) $NE(G) \cap TE(G) \neq \emptyset$;
- (ii) $NEP(G) = TEP(G)$.

The prisoner's dilemma is a game of type I.

The next definition plays a crucial role.

DEFINITION 3.6 $(\tilde{s}_1, \tilde{s}_2)$ is a saddle point of G if and only if for all $(s_1, s_2) \in S_1 \times S_2$,

$$u_1(s_1, \tilde{s}_2) \leq u_1(\tilde{s}_1, \tilde{s}_2) \leq u_1(\tilde{s}_1, s_2) \quad (6)$$

and

$$u_2(\tilde{s}_1, s_2) \leq u_2(\tilde{s}_1, \tilde{s}_2) \leq u_2(s_1, \tilde{s}_2) \quad (7)$$

We denote $S(G)$ the set of saddle points of G and $SP(G)$ the set of saddle point payoffs.

The following property shows that condition (i) is equivalent to $S(G) \neq \emptyset$.

PROPERTY 3.2 For every two-person game G,

1. $S(G) = NE(G) \cap TE(G)$.
2. If $NEP(G) \cap TEP(G)$ is non empty, then $NEP(G) \cap TEP(G) = \{v\} = \{(\text{Val} u_i)_{i \in N}\}$. Moreover, if $S(G)$ is non empty, the infimum and the supremum are attained in (1).

Proof.

1) We show the first assertion:

$$\begin{aligned} \tilde{s} \in NE(G) \cap TE(G) &\Leftrightarrow \forall i \in N, s_i \in S_i, s_{-i} \in S_{-i}, u_i(s_i, \tilde{s}_{-i}) \leq u_i(\tilde{s}) \leq u_i(\tilde{s}_i, s_{-i}) \\ &\Leftrightarrow \tilde{s} \in S(G) \end{aligned}$$

2) Let $e \in NEP(G) \cap TEP(G)$. Let $s^* \in NE(G)$ and $\bar{s} \in TE(G)$ such that:

$$e_i = u_i(s^*) = u_i(\bar{s}) \text{ for all } i \in N.$$

By definition of a Nash equilibrium, $e_i = u_i(s^*) = \sup_{S_i} u_i(s_i, s_{-i}^*) \geq \inf_{S_{-i}} \sup_{S_i} u_i(s_i, s_{-i})$

By definition of a twisted equilibrium, $e_i = u_i(\bar{s}) = \inf_{S_{-i}} u_i(\bar{s}_i, s_{-i}) \leq \sup_{S_i} \inf_{S_{-i}} u_i(s_i, s_{-i})$.

Since $\inf_{S_{-i}} \sup_{S_i} u_i(s_i, s_{-i}) \geq \sup_{S_i} \inf_{S_{-i}} u_i(s_i, s_{-i})$, we get $e_i = \text{Val} u_i$.

If $S(G)$ is non empty, (6) and (7) imply $v_i = \min_{S_{-i}} \max_{S_i} u_i(s_i, s_{-i}) = \max_{S_i} \min_{S_{-i}} u_i(s_i, s_{-i})$

for all $i \in N$. ■

The following definition is in the spirit of Aumann's definition of good strategies [1].

DEFINITION 3.7 $s = (s_i, s_{-i})$ is robust if there exists $w \in \mathbb{R}^2$ such that for each $i \in N$ when player i plays s_i , whatever player $-i$ plays:

- a) player i receives at least w_i and
- b) player $-i$ receives at most w_{-i} .

We denote the set of robust strategies of G , $R(G)$.

Let s be a couple of robust strategies. s_i guarantees w_i to player i and s_{-i} guarantees that player i receives at most w_i , so $w_i = v_i = \text{Val}u_i$. We say that s_i is robust for player i if for every strategy s_{-i} of player $-i$:

- (a) $u_i(s_i, s_{-i}) \geq v_i$;
- (b) $u_{-i}(s_i, s_{-i}) \leq v_{-i}$.

PROPERTY 3.3 For every two-person game G , $S(G) = R(G)$.

Indeed, if s is robust, $u_i(s) = v_i$, and thanks to (6) and (7) every couple of robust strategies is a saddle point. Conversely, every saddle point is robust.

PROPERTY 3.4 The saddle points are interchangeable.

Proof.

Let $\tilde{s}, \tilde{s}' \in S(G)$. $\tilde{s}_1$ and $\tilde{s}'_1$ (resp. $\tilde{s}_2$ and $\tilde{s}'_2$) are robust for player 1 (resp. player 2), then $(\tilde{s}_1, \tilde{s}'_2)$ and $(\tilde{s}'_1, \tilde{s}_2)$ are robust, and so are saddle points. ■

For every game of type B , the set of Nash equilibria and the set of twisted equilibria coincide, so that every game of type B is of type I. The converse is false.

EXAMPLE 3.4

$$G_4 = \begin{array}{cc} & \begin{matrix} A_2 & B_2 \end{matrix} \\ \begin{matrix} A_1 \\ B_1 \end{matrix} & \begin{pmatrix} 2, 2 & 3, 1 \\ 1, 3 & 2, 0 \end{pmatrix} \end{array}$$

(A_1, A_2) is the unique Nash equilibrium of G_4 . It is also the unique twisted equilibrium; thus G_4 is of type I. On the other hand, it is not a game of type C (and then not a game of type B) because $u_i(A_1, B_2) > u_i(B_1, B_2)$ for every $i \in N$.

G_4 is a game of type I but not of type C . One could ask whether type C implies type I. Example 3.3 shows that this is not the case: when player i minimizes the payoff of player $-i$, he is indifferent between A_i and B_i , then the set of twisted equilibria is equal to the product of the strategies sets. G_3 verifies condition (i) of the definition 3.5 but not condition (ii).

3.6 GAME OF TYPE II

Now, we introduce a new class of games by weakening condition (ii) of definition 3.5.

DEFINITION 3.8 *G is a game of type II if it satisfies the two following conditions:*

- (i) $S(G) \neq \emptyset$;
- (ii') $NEP(G) \subset TEP(G)$.

Every game of type I is of type II. The converse is not true, as example 3.3 shows: we saw that the set of twisted equilibria of G_3 is the product of the strategy sets, so G_3 verifies condition (ii').

A game of type C may not be of type I. We show in section 4 that every game of type C is of type II.

3.7 GAME OF TYPE III

DEFINITION 3.9 *G is a game of type III if $S(G) \neq \emptyset$.*

Every game of type II is of type III. The converse is false.

EXAMPLE 3.5

$$G_5 = \begin{array}{cc} & \begin{matrix} A_2 & B_2 \end{matrix} \\ \begin{matrix} A_1 \\ B_1 \end{matrix} & \begin{pmatrix} 2, 2 & 0, 0 \\ 0, 0 & 1, 1 \end{pmatrix} \end{array}$$

$$NE(G_5) = \{(A_1, A_2); (B_1, B_2); ((\frac{1}{3}, \frac{2}{3}), (\frac{1}{3}, \frac{2}{3}))\}, TE(G_5) = \{(A_1, B_2); (B_1, A_2); ((\frac{1}{3}, \frac{2}{3}), (\frac{1}{3}, \frac{2}{3}))\}.$$

We have $NE(G_5) \cap TE(G_5) \neq \emptyset$ but $NEP(G_5) \not\subset TEP(G_5)$.

3.8 GAME OF TYPE IV

DEFINITION 3.10 *G is a game of type IV if $NEP(G) \subset TEP(G)$.*

If a game is of type II, then it is of type IV, but the following example shows that the converse is not true:

EXAMPLE 3.6

$$G_6 = \begin{array}{cc} & \begin{matrix} A_2 & B_2 \end{matrix} \\ \begin{matrix} A_1 \\ B_1 \end{matrix} & \begin{pmatrix} 2, 0 & 0, 1 \\ 0, 1 & 1, 0 \end{pmatrix} \end{array}$$

$NE(G_6) = \{((\frac{1}{2}, \frac{1}{2}); (\frac{1}{3}, \frac{2}{3}))\}$, $TE(G_6) = \{((\frac{1}{3}, \frac{2}{3}); (\frac{1}{2}, \frac{1}{2}))\}$ and $NEP(G_6) = TEP(G_6) = \{v\} = \{(\frac{2}{3}, \frac{1}{2})\}$. G_6 is a game of type IV but not of type II because G_6 has no saddle point.

Remarks.

1. Examples G_5 and G_6 show that conditions (i) and (ii') are independent from each other.

2. The equilibrium payoff of player 1 is equal to v_1 . But the optimal strategy of player 1 in the zero-sum game where his payoff is given by his payoff in G_6 is different from his Nash equilibrium strategy [4].

4 PROPERTIES

The following property is a consequence of property 3.2 and condition (ii'):

PROPERTY 4.1 *For every game G of type IV, $NEP(G) = \{v\} = \{(Val u_i)_{i \in N}\}$.*

Property 4.1 is false for games of type III as example 3.5 shows.

PROPERTY 4.2 *For every game G of type C, $S(G) = NE(G)$.*

Proof.

Let G be a game of type C. According to property 3.2, $S(G) \subset NE(G)$.

Let $s^* \in NE(G)$. For every $(s_1, s_2) \in S_1 \times S_2$,

$$u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^*) \quad (8)$$

and

$$u_2(s_1^*, s_2^*) \geq u_2(s_1^*, s_2) \quad (9)$$

In (9), there are two cases:

1. $u_2(s_1^*, s_2^*) > u_2(s_1^*, s_2) \xrightarrow{G} u_1(s_1^*, s_2^*) \leq u_1(s_1^*, s_2)$;
2. $u_2(s_1^*, s_2^*) = u_2(s_1^*, s_2) \xrightarrow{G} u_1(s_1^*, s_2^*) = u_1(s_1^*, s_2)$;

We have $u_1(s_1^*, s_2) \geq u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^*)$.

We proceed similarly for player 2. ■

We obtain the following property, already proved by Kats and Thisse [9].

PROPERTY 4.3 *The Nash equilibria of a game of type C are interchangeable.*

Proof.

This property is a consequence of properties 3.4 and 4.2. ■

PROPERTY 4.4 *Every game of type C is of type II.*

Proof.

Let G be a game of type C. According to property 4.2, $S(G) = NE(G)$; then $S(G)$ is non empty and G verifies condition (i) of definition 3.8.

We have $S(G) = NE(G) = NE(G) \cap TE(G)$, then $NE(G) \subset TE(G)$ and G satisfies condition (ii'). ■

A game of type II is not necessarily of type C , as example 3.4 shows¹

PROPERTY 4.5 *For every two-person game G , if for all $i \in N$, S_i is convex, u_i is quasi-concave in s_i and quasi-convex in s_{-i} , then $S(G)$ is convex.*

Proof.

Let $\tilde{s}, \tilde{s}' \in S(G)$. We note $\tilde{s}_i^\lambda = \lambda \tilde{s}_i + (1 - \lambda) \tilde{s}'_i$, $0 \leq \lambda \leq 1$. From now on, we use the characterization: let $f : X \rightarrow \mathbb{R}$. f is quasi-convex if and only if:

$$\forall x, x' \in X, \forall \lambda \in [0, 1], f(\lambda x + (1 - \lambda)x') \leq \max\{f(x), f(x')\} \quad (10)$$

Step 1: $(\tilde{s}_1, \tilde{s}_2^\lambda)$ and $(\tilde{s}_1^\lambda, \tilde{s}_2)$ are saddle points.

We have to show that $(\tilde{s}_1, \tilde{s}_2^\lambda)$ are robust strategies for all λ . Since $\tilde{s}_1$ is robust for player 1, we show that $\tilde{s}_2^\lambda$ is robust for player 2. Let $s_1 \in S_1$.

- a) By quasi-concavity of u_2 in s_2 and (10), we have:
 $u_2(s_1, \tilde{s}_2^\lambda) \geq \min\{u_2(s_1, \tilde{s}_2), u_2(s_1, \tilde{s}'_2)\}$, so $u_2(s_1, \tilde{s}_2^\lambda) \geq v_2$ (because $\tilde{s}_2$ and $\tilde{s}'_2$ are robust for player 2).
- b) By quasi-convexity of u_1 in s_2 and (10), we have:
 $u_1(s_1, \tilde{s}_2^\lambda) \leq \max\{u_1(s_1, \tilde{s}_2), u_1(s_1, \tilde{s}'_2)\} \leq v_1$.

So, $\tilde{s}_2^\lambda$ is robust for player 2. We show similarly that $(\tilde{s}_1^\lambda, \tilde{s}_2)$ is a saddle point.

Step 2: $\tilde{s}^\lambda$ is a saddle point.

As the saddle points are interchangeable and as $(\tilde{s}_1, \tilde{s}_2^\lambda)$ and $(\tilde{s}_1^\lambda, \tilde{s}_2)$ are saddle points, $(\tilde{s}_1^\lambda, \tilde{s}_2^\lambda)$ is a saddle point. ■

In fact, for a game of type C , if u_1 is quasi-concave in s_1 , then u_2 is quasi-convex in s_1 .

Indeed, let $G = (S_1, S_2, u_1, u_2)$ be a game of type C where u_1 is quasi-concave in s_1 . We want to show that u_2 verifies (10) in s_1 . We fix s_2 . Let $s_1, s'_1 \in S_1$. Let $\tilde{s}_1 \in \arg \min\{u_1(s_1, s_2), u_1(s'_1, s_2)\}$. Then, by definition of a game of type C , $\tilde{s}_1 \in \arg \max\{u_2(s_1, s_2), u_2(s'_1, s_2)\}$. We have:

$$\begin{cases} u_1(s_1, s_2) \geq u_1(\tilde{s}_1, s_2) & \Rightarrow & u_2(s_1, s_2) \leq u_2(\tilde{s}_1, s_2) \\ u_1(s'_1, s_2) \geq u_1(\tilde{s}_1, s_2) & \Rightarrow & u_2(s'_1, s_2) \leq u_2(\tilde{s}_1, s_2) \end{cases}$$

But u_1 is quasi-concave in s_1 , then $u_1(s_1^\lambda, s_2) \geq u_1(\tilde{s}_1, s_2)$, which implies $u_2(s_1^\lambda, s_2) \leq u_2(\tilde{s}_1, s_2)$.

We have shown that for all $s_1, s'_1 \in S_1$, $u_2(s_1^\lambda, s_2) \leq \max\{u_2(s_1, s_2), u_2(s'_1, s_2)\}$. We get the following property [9].

PROPERTY 4.6 *For every game of type C such that S_i is convex and u_i is quasi-concave in s_i for every $i \in N$, the set of Nash equilibria is convex.*

1. Figure 1 in appendix shows the connections between the different classes.

5 CORRELATED EQUILIBRIUM

5.1 DEFINITIONS

Let $g = (S_1, S_2, u_1, u_2)$ be a finite game, i.e. a game where the set of strategies of each player is finite. $G = (\Delta^{S_1}, \Delta^{S_2}, u_1, u_2)$ denotes its mixed extension (for $t = (t_1, t_2) \in \Delta^{S_1} \times \Delta^{S_2}$, $u_i(t) = \sum_{s_1 \in S_1, s_2 \in S_2} t_1(s_1)t_2(s_2)u_i(s_1, s_2)$).

Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let θ_i be a measurable function from $(\Omega, \mathcal{A}, P)$ to a finite set of signals of player i , X_i . $\gamma = ((\Omega, \mathcal{A}, P), (\theta_i)_{i \in N})$ is called *correlation device*. Given a game G and a device γ , we define a new game named G_γ as follows: at stage 1, $\omega \in \Omega$ is chosen according to P . $\theta_i(\omega)$ is announced to player i . At the next stage, every player plays in G .

A strategy of player i in G_γ is a mapping $\sigma_i : \Omega \rightarrow \Delta^{S_i}$ such that $\sigma_i(\omega) = \sigma_i(\omega')$ if $\theta_i(\omega) = \theta_i(\omega')$.

The payoff of player i in G_γ is:

$$\phi_i(\sigma_1, \sigma_2) = \int_{\Omega} u_i(\sigma_1(\omega), \sigma_2(\omega)) P(d\omega)$$

DEFINITION 5.1 [2] *A Nash equilibrium of G_γ is a correlated equilibrium of G . $CEP(G)$ denotes the set of correlated equilibrium payoffs of G obtained when the mechanism γ varies: $CEP(G) = \cup_{\gamma} NEP(G_\gamma)$.*

We are interested in the distribution on $S_1 \times S_2$ induced by a correlated equilibrium: if $\sigma = (\sigma_1, \sigma_2)$ is a couple of strategies, $\mu_{\sigma(\omega)}(s_1, s_2) = \sigma_1(\omega)(s_1)\sigma_2(\omega)(s_2)$ is the probability of (s_1, s_2) if $\sigma(\omega)$ is played. The probability distribution on S induced by σ is $Q_\sigma = (Q_\sigma(s_1, s_2))_{s_1 \in S_1, s_2 \in S_2}$:

$$Q_\sigma(s_1, s_2) = \int_{\Omega} \mu_{\sigma(\omega)}(s_1, s_2) P(d\omega)$$

$CED(G)$ denotes the set of distributions Q such that there exists a correlation device γ and an equilibrium of G_γ inducing Q .

We recall the notion of canonical correlated equilibrium [3, 18].

DEFINITION 5.2

1. *A canonical correlation device is a correlation device such that $\Omega = S_1 \times S_2$, $\theta_i(\omega) = s_i$.*
2. *A canonical correlated equilibrium is a correlated equilibrium such that the correlation device is canonical and such that the equilibrium strategy of each player consists in playing the signal he received.*

THEOREM 5.1 *The set of correlated equilibrium distributions coincides with the set of canonical correlated equilibrium distributions.*

Let Q be a correlated equilibrium distribution. According to Theorem 5.1, Q can be induced by a canonical equilibrium. If player i received the signal $\theta_i(\omega) = s_i$, he has no incentive to deviate in playing s'_i , then we have the characterization:

PROPERTY 5.1 $CED = \{Q \in \Delta^{S_1 \times S_2} : \forall i \in N, \forall s_i, s'_i \in S_i,$

$$\sum_{s_{-i} \in S_{-i}} Q(s_{-i}|s_i) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} Q(s_{-i}|s_i) u_i(s'_i, s_{-i})\}$$

5.2 CORRELATED EQUILIBRIUM PAYOFF

Let $g = (S_1, S_2, u_1, u_2)$ be a finite game. We show that if the mixed extension G of g is of type C , then $v = (\min_{t_{-i} \in \Delta^{S_{-i}}} \max_{t_i \in \Delta^{S_i}} u_i(t_i, t_{-i}))_{i \in N}$ is the unique correlated equilibrium payoff.

Remark.

De Wolf [5] has proved independently that every finite game whose mixed extension is of type C has a unique correlated equilibrium payoff.

THEOREM 5.2 *For every finite game whose mixed extension is of type C , the correlated equilibrium payoff is unique and equal to v .*

Proof.

Let $g = (S_1, S_2, u_1, u_2)$ be a game whose mixed extension is of type C .

Every correlated equilibrium payoff of player i is greater than v_i . We show that every correlated equilibrium payoff of player 2 is smaller than v_2 .

Fix $s_1 \in S_1$. Let Q be a correlated equilibrium distribution. According to property 5.1, we have for all $s'_1 \in S_1$,

$$\sum_{s_2 \in S_2} Q(s_2|s_1) u_1(s_1, s_2) \geq \sum_{s_2 \in S_2} Q(s_2|s_1) u_1(s'_1, s_2) \quad (11)$$

$u_1(s_1, s_2)$ is linear in s_2 , so (11) becomes

$$u_1(s_1, t_2(s_1)) \geq u_1(s'_1, t_2(s_1)) \quad (12)$$

where $t_2(s_1) = \{Q(s_2|s_1)\}_{s_2 \in S_2} \in \Delta^{S_2}$. But G is of type C so (12) implies that for all $s'_1 \in S_1$, $u_2(s_1, t_2(s_1)) \leq u_2(s'_1, t_2(s_1))$. This implies:

$$u_2(s_1, t_2(s_1)) \leq u_2(t_1, t_2(s_1)) \quad \forall t_1 \in \Delta^{S_1} \quad (13)$$

But there exists a strategy $\bar{t}_1$ of player 1 such that $u_2(\bar{t}_1, t_2(s_1)) \leq v_2$. From (13) with $t_1 = \bar{t}_1$ and from the linearity of $u_2(s_1, s_2)$ in s_2 , we get for all $s_1 \in S_1$,

$$\sum_{s_2 \in S_2} Q(s_2|s_1) u_2(s_1, s_2) \leq v_2$$

which implies:

$$\sum_{s_1 \in S_1, s_2 \in S_2} Q(s_1, s_2) u_2(s_1, s_2) \leq v_2 \quad (14)$$

which shows that the correlated equilibrium payoff of player 2 is smaller than v_2 . We proceed similarly for player 1. ■

Remark.

Theorem 5.2 is still valid if we assume only that the mixed extension G verifies (4) and (5) for all i in N , for all s_{-i} in $\Delta^{S_{-i}}$ and for all s_i, s'_i only in S_i .

Theorem 5.2 is no longer true if we assume that only g , and not its mixed extension, is of type C :

EXAMPLE 5.1

$$(P_1, P_2) = \begin{matrix} & \begin{matrix} A_2 & B_2 & C_2 \end{matrix} \\ \begin{matrix} A_1 \\ B_1 \\ C_1 \end{matrix} & \begin{pmatrix} 4, 0 & 1, 1 & 0, 4 \\ 1, 1 & 1, 1 & 1, 1 \\ 0, 4 & 1, 1 & 4, 0 \end{pmatrix} \end{matrix}$$

The game $g = (S_1, S_2, P_1, P_2)$, with $S_i = \{A_i, B_i, C_i\}$, is a game of type C . Its unique Nash equilibrium payoff is $(1, 1)$. But $((\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2}))$ is a mixed Nash equilibrium, inducing a correlated equilibrium distribution which gives a payoff of two to each player.

Rosenthal [16] gave a counter-example of Theorem 5.2 when the mixed extension is of type I:

EXAMPLE 5.2

$$\begin{matrix} & \begin{matrix} A_2 & B_2 & C_2 & D_2 \end{matrix} \\ \begin{matrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{matrix} & \begin{pmatrix} 6, 6 & 2, 7 & -1, 6.5 & -1, 1 \\ 7, 2 & 0, 0 & -1, 1.4 & -1, 3 \\ 6.5, -1 & 1.4, -1 & 0, 0 & 0, 0 \\ 1, -1 & 3, -1 & 0, 0 & 0, 0 \end{pmatrix} \end{matrix}$$

Every randomization over C_i and D_i is a Nash and a twisted equilibrium, and there are no other twisted and Nash equilibria. The game is of type I, and its equilibrium payoff is equal to $(0, 0)$. But the distribution Q defined below is a correlated equilibrium distribution which gives 5 to each player:

$$Q = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Is the converse of Theorem 5.2 true, *i.e.* is a game which has a unique correlated equilibrium payoff equal to the threat point of type C ? The answer is negative, as example 3.6 shows: G_6 is not of type C because player 2 is indifferent between $((\frac{1}{2}, \frac{1}{2}), A_2)$ and $((\frac{1}{2}, \frac{1}{2}), B_2)$ but not player 1. We assume that there is a correlated

equilibrium distribution Q which gives strictly more than a Nash equilibrium to one of the two players. This implies that $Q(1, 1) > \frac{1}{6}$. The equilibrium conditions imply: $2Q(1, 1) \geq Q(1, 2)$, $Q(2, 2) \geq 2Q(2, 1)$, $Q(2, 1) \geq Q(1, 1)$ and $Q(1, 2) \geq Q(2, 2)$. Hence:

$$2Q(1, 1) \geq Q(1, 2) \geq Q(2, 2) \geq 2Q(2, 1) \geq 2Q(1, 1)$$

and $\sum_{s_1, s_2} Q(s_1, s_2) \geq 6Q(1, 1) > 1$ which is impossible.

We extend Theorem 5.2 to a larger class of games. We denote BR_i the best-response correspondence of player i : $BR_i(s_{-i}) = \arg \max_{S_i} u_i(\cdot, s_{-i})$.

DEFINITION 5.3 [16] $G = (S_1, S_2, u_1, u_2)$ is best-response equivalent to a zero-sum game if there exists a zero-sum game Γ such that for each player $i \in N$, the best-response correspondence of player i in G is equal to his best-response correspondence in Γ .

PROPERTY 5.2 [5] Let G be a game which is best-response equivalent to a zero-sum game. Then:

1. $NE(G)$ is convex;
2. $NE(G) = NE_1(G) \times NE_2(G)$, where $NE_i(G) = \{s_i : \exists s_{-i}, (s_i, s_{-i}) \in NE(G)\}$;
3. Q is a correlated equilibrium distribution if and only if for all $i \in N$, for all $s_i \in S_i$ such that $Q(s_i) > 0$, $\{Q(s_{-i}|s_i)\}_{s_{-i}} \in NE_{-i}$.

A game which is best-response equivalent to a zero-sum game may have several equilibrium payoffs. But, by adding a constraint on the best-response correspondence, we get uniqueness of equilibrium payoff.

DEFINITION 5.4 For every $i \in N$, we define:

$$\underline{BR}_i(s_{-i}) = \{s_i \in S_i : s_i \in BR_i(s_{-i}) \text{ and } s_i \in \arg \min_{\tilde{s}_i \in S_i} u_{-i}(\tilde{s}_i, s_{-i})\}$$

Namely s_i belongs to $\underline{BR}_i(s_{-i})$ if and only if:

- i) s_i is a best response to s_{-i} ;
- ii) s_i minimizes player $-i$'s payoff.

THEOREM 5.3 Let $g = (S_1, S_2, u_1, u_2)$ be a finite game such that its mixed extension is:

- a) best-response equivalent to a zero-sum game;
- b) for all $i \in N$, for all $s_{-i} \in NE_{-i}(G)$, $BR_i(s_{-i}) = \underline{BR}_i(s_{-i})$;

then v is its unique correlated equilibrium payoff.

Proof.

Let G be a game that verifies a) and b). The strategy $t_2(s_1)$, defined in the proof of theorem 5.2, is an element of $NE_2(G)$ (from condition (a) and property 5.2).

Equation (12) implies:

$$s_1 \in BR_1(t_2(s_1)) \quad (15)$$

From (b) and (15), we get

$$s_1 \in \underline{BR}_1(t_2(s_1))$$

and (13). ■

Every game of type C verifies conditions a) and b). The converse is not true: the game G_4 verifies conditions (a) and (b) but is not of type C .

Remarks.

1. From the proof of theorem 5.2, we deduce that every game such that “for all $i \in N$, $\forall s_{-i} \in S_{-i}$, $BR_i(s_{-i}) = \underline{BR}_i(s_{-i})$ ” has a unique correlated equilibrium payoff which is equal to v .
2. The example below, due to Moulin and Vial [15], shows that the property “ $NEP(G) = \{v\}$ ” does not imply the property “ $CEP(G) = \{v\}$ ”:

EXAMPLE 5.3

$$\begin{array}{c|ccc} & A_2 & B_2 & C_2 \\ \hline A_1 & (0, 0) & (1, 2) & (2, 1) \\ B_1 & (2, 1) & (0, 0) & (1, 2) \\ C_1 & (1, 2) & (2, 1) & (0, 0) \end{array}$$

$((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}); (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))$ is the unique Nash equilibrium of this game, and it gives his threat point to every player. But the distribution Q defined below is a correlated equilibrium distribution which gives each player strictly more than his threat point v_i :

$$Q = \begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 \end{pmatrix}$$

6 ANTAGONISM IN PURE AND IN MIXED STRATEGIES

6.1 GAME OF TYPE A

Two games are equivalent if they have the same set of strategies and the same ordinal preferences. Leininger, Thorlund-Petersen and Weibull showed that every game of type A (in pure strategies) is equivalent to a zero-sum game [13].

The notion of game of type A can be extended to mixed strategies: a game is of type A in mixed strategies if the mixed extension of this game is of type A. These two notions are

not equivalent: a game may be of type A in pure strategies but not in mixed strategies. Leininger, Thorlund-Petersen and Weibull have characterized a game of type A in mixed strategies [13]:

PROPOSITION 6.1 *G is a game of type A in mixed strategies if and only if there exist two positive real numbers α and β such that for all $(s_1, s_2) \in S_1 \times S_2$:*

$$\alpha u_1(s_1, s_2) + \beta u_2(s_1, s_2) = \text{constant}$$

6.2 GAME OF TYPE C

Is it possible to restrict conditions (4) and (5) to pure strategies while keeping properties of equilibria in mixed strategies from section 4? The answer is no. Example 5.1 shows that a game of type C in pure strategies can have a mixed equilibrium which gives strictly more than the pure equilibrium to each player.

7 REPEATED GAMES

7.1 NASH EQUILIBRIUM

Let $G = (S_1, S_2, u_1, u_2)$ be a finite two-person game. We associate to G the n times repeated games G^n played as follows: at each stage $t \in \{1, \dots, n\}$, player i chooses an action $s_i^t \in S_i$. $s^t = (s_1^t, s_2^t)$ is announced to both players.

A strategy of player i describes the way player i chooses his action at each stage of the repeated game: it is a sequence $\sigma_i = (\sigma_i^t)_{t=1}^n$ where $\sigma_i^t : S^{t-1} \rightarrow \Delta^{S_i}$.

A couple of strategies $\sigma = (\sigma_1, \sigma_2)$ induces a probability distribution over S^n . It is denoted by P and E_P stands for the expectation induced by P .

The payoff of player i in G^n is $g_i^n(\sigma) = \frac{1}{n} E_P[\sum_{t=1}^n u_i(s^t)]$.

We denote $NEP^n(G)$ the set of Nash equilibrium payoffs of G^n .

PROPERTY 7.1 *For every game G of type IV, for all $n \in \mathbb{N}^*$, $NEP^n(G) = \{v\}$.*

Property 7.1 is a consequence of:

PROPOSITION 7.1 [17] *For every game G such that $NEP^1(G) = \{v\}$, $NEP^n(G) = \{v\}$ for all $n \in \mathbb{N}^*$.*

7.2 COMMUNICATION EQUILIBRIUM

The aim of this section is to extend Proposition 7.1 to more general notions of equilibrium. Again, we assume that the players are involved in a repeated game. Before playing, at each round of the game, the players can communicate: at stage $t \geq 1$, player

k sends an input from a set I_k^t . The mechanism selects an output from the set $O_1^t \times O_2^t$ according to a probability μ^t . $o_k^t \in O_k^t$ is announced to player k . This leads to:

DEFINITION 7.1 [7] *A communication device of G^n is a collection $\{(I_k^t)_{k \in N}, (O_k^t)_{k \in N}, \mu^t\}_{t=1, \dots, n}$ where I_k^t is the set of inputs of player k at stage t , O_k^t is his set of outputs and μ^t is a transition probability on the outputs.*

A correlation device is a device such that O_k^t is a singleton for all $t \geq 2$ and all $k \in N$; and I_k^t is a singleton for all $t \geq 1$ and all $k \in N$.

Given a communication device d , we define the extended game G_d^n of G_n by adding d to G . Its progress is the following: each stage t may be splitted into three parts: first, player k sends an input $i_k^t \in I_k^t$. Then the device selects an output $o^t = (o_k^t)_{k \in N}$ according to the probability μ^t . o_k^t is announced to player k . At last the players choose an action as in G^n .

A *communication equilibrium* of G^n is a Nash equilibrium of G_d^n for some device d [7]. CEP^{*n} (resp. CEP^n) denotes the set of communication (resp. correlated) equilibrium payoffs.

DEFINITION 7.2 *A communication device is canonical if I_k^t is the set of information of player k at stage t and $O_k^t = S_k$.*

A correlation device is canonical if O_k^1 is the set of strategies of player k .

THEOREM 7.1 [7] *CEP^{*n} (resp. CEP) is equal to the set of canonical communication (resp. canonical correlated) equilibrium payoffs.*

PROPOSITION 7.2 *If $CEP^1 = \{v\}$, then $CEP^n = \{v\}$ for all $n \in \mathbb{N}^*$.*

Proof.

Let G be a game such that $CEP^1 = \{v\}$. Let σ be a canonical correlated equilibrium of G^n .

1. At stage n , for all history h^n such that $P_\sigma(h^n) > 0$, $\sigma^n(h^n)$ is an equilibrium of G_1 , so that each player receives v_i as a payoff.
2. Let p be the greatest integer such that there exists a history h^p such that $\sigma^p(h^p)$ induces a payoff strictly greater than v_i to player i ; $p < n$. After h^p , a player j has a profitable deviation from $\sigma_j^p(h^p)$, which is impossible. ■

PROPOSITION 7.3 *If $CEP^{*1} = \{v\}$, then $CEP^{*n} = \{v\}$ for all $n \in \mathbb{N}^*$.*

Proof.

Let G be a game such that $CEP^{*1} = \{v\}$. Let σ be a canonical communication equilibrium of G^n .

1. Assume that there exists a history h^n such that $P_\sigma(h^n) > 0$ and such that $\sigma^n(h^n)$ induces to player i a payoff strictly greater than v_i at stage n . Consider a device d' of G^1 defined by $I_j^1 = \{s_j^{n-1}\}$, $O_j^1 = \{s_j^n\}$ for all $j \in N$ and $P^1 = P^n$. Then the

profile $\alpha = \sigma^n(h^n)$ is an equilibrium of G_d^1 , which induces a payoff strictly greater than v_i to player i . This contradicts $CEP^{*1} = \{v\}$.

2. Let p be the greatest integer such that there exists a history h^p such that $\sigma^p(h^p)$ induces a payoff strictly greater than v_i to a player i ; $p < n$. After h^p , a player j has a profitable deviation from $\sigma_j^p(h^p)$, which can no longer be punished. Player j has thus a profitable deviation, which is impossible. ■

8 SYMMETRIC GAMES

DEFINITION 8.1 A game $G = (S_1, S_2, u_1, u_2)$ is symmetric if $S_1 = S_2 = S$ and for all $s_1 \in S_1, s_2 \in S_2, u_1(s_1, s_2) = u_2(s_2, s_1)$.

PROPERTY 8.1 Every symmetric game of type C verifies

1. Every player has the same set of equilibrium strategies.
2. Every player has the same equilibrium payoff.

We define the function **Sgn** from $\mathbb{R}$ to $\{-1, 0, 1\}$ by $\mathbf{Sgn}(x) = 1$ if $x > 0$, $\mathbf{Sgn}(x) = 0$ if $x = 0$ and $\mathbf{Sgn}(x) = -1$ if $x < 0$.

We denote by $\mathcal{F}$ the set of applications f from $\mathbb{R}$ to $\mathbb{R}$ such that:

- i) f is increasing
- ii) $f(-x) = -f(x)$ for all $x \in \mathbb{R}$,
- iii) $f(x) = 0$ if and only if $x = 0$.

PROPERTY 8.2 Let $G = (S_1, S_2, u_1, u_2)$ be a symmetric game of type C , Ψ an element of $\mathcal{F}$ and $\Gamma = (S_1, S_2, \Psi(u_1 - u_2), -\Psi(u_1 - u_2))$, where $\Psi(u_1 - u_2)(s_1, s_2) = \Psi(u_1(s_1, s_2) - u_2(s_1, s_2))$. Then $NE(G) = NE(\Gamma)$.

Proof.

We show first the following lemma, true for all game of type C :

LEMMA 1 Let $G = (S_1, S_2, u_1, u_2)$ be a game of type C and $G' = (S_1, S_2, u_1 - u_2, u_2 - u_1)$. Then $NE(G) = NE(G')$.

Proof of the Lemma.

Let $i \in N$ and $s^* = (s_1^*, s_2^*)$ be a Nash equilibrium of G . We have $u_i(s^*) \geq u_i(s_i, s_{-i}^*)$ for all $s_i \in S_i$. This implies $u_{-i}(s^*) \leq u_{-i}(s_i, s_{-i}^*)$, and $(u_i - u_{-i})(s^*) \geq (u_i - u_{-i})(s_i, s_{-i}^*)$ for every $s_i \in S_i$. So $NE(G) \subset NE(G')$.

Let s^* be an equilibrium of G' . We show that s^* is an equilibrium of G . Otherwise, there exist $i, \bar{s}_i$ such that $u_i(\bar{s}_i, s_{-i}^*) > u_i(s^*)$. This implies $u_{-i}(\bar{s}_i, s_{-i}^*) \leq u_{-i}(s^*)$, and $(u_i - u_{-i})(\bar{s}_i, s_{-i}^*) > (u_i - u_{-i})(s^*)$ which is impossible. ■

We want to show that $NE(G') = NE(\Gamma)$. It is clear that $NE(G') \subset NE(\Gamma)$. Let $s^* \in NE(\Gamma)$. Γ is a symmetric zero-sum game ($\Psi(u_1 - u_2)(s_1, s_2) = \Psi(u_1(s_1, s_2) -$

$u_2(s_1, s_2)) = \Psi(u_2(s_2, s_1) - u_1(s_2, s_1)) = -\Psi(u_1 - u_2)(s_2, s_1))$. So we have $\Psi(u_i - u_{-i})(s_i, s_{-i}^*) \leq 0$ for all s_i , which implies:

$$(u_i - u_{-i})(s_i, s_{-i}^*) \leq 0 = (u_i - u_{-i})(s^*) \text{ and so } s^* \in NE(G') \quad \blacksquare$$

Especially, when $\Psi = \text{Sgn}$, at equilibrium, it is equivalent to maximize his payoff or to have a payoff better than his opponent.

Remarks.

1. The symmetry hypothesis is necessary, as the following example shows: $\Psi = \text{Sgn}$, $S_1 = S_2 = S = [0, 1]$, $u_1(s_1, s_2) = 3s_1s_2 - 2s_1 - 2s_2 + 2$, $u_2 = -u_1$. $G = (S_1, S_2, u_1, u_2)$ is a game of type C , non symmetric. We have $\Psi(u_1 - u_2)(s_1, s_2) = \text{Sgn}(s_1(3s_2 - 2) + 2(1 - s_2))$. $(\frac{1}{2}, \frac{1}{2})$ is an equilibrium of Γ but not of G .
2. The hypothesis “ $\Psi(x) = 0$ if and only if $x = 0$ ” is necessary: $S_1 = S_2 = [0, 1]$, $u_1(s_1, s_2) = s_1 - s_2$, $u_2 = -u_1$, $\Psi(x) = 0$ if $x \in [-1, 1]$, else $\Psi(x) = x$. Then $\Psi(u_1 - u_2)(s_1, s_2) = \Psi(2(s_1 - s_2))$. $(\frac{1}{2}, \frac{1}{2})$ is an equilibrium of Γ but not of G .
3. Property 8.2 is a generalization of a result of Laslier [12], proved for zero-sum games and with $\Psi = \text{Sgn}$.

9 EXAMPLES

9.1 BERTRAND'S MODEL

We consider two firms producing a same commodity and having the same cost function (see for example [10]). We denote by c the marginal cost. The firm i chooses a price $p_i \geq c$. $D(p)$ denotes the demand function of the consumer, and $D_i(p_i, p_{-i})$ is the demand for firm i :

$$D_i(p_i, p_{-i}) = \begin{cases} D(p_i) & \text{if } p_i < p_{-i} \\ D(p_i)/2 & \text{if } p_i = p_{-i} \\ 0 & \text{else} \end{cases}$$

π_i is the profit of firm i , $\pi_i(p_i, p_{-i}) = (p_i - c)D_i(p_i, p_{-i})$.

At equilibrium, each firm chooses its marginal cost c for the price, and has a profit equal to zero [10]. At twisted equilibrium, every firm minimizes the profit of the other firm. To choose a price equal to c gives a profit 0 to the other firm, so (c, c) is a twisted equilibria, too. $(0, 0)$ is the unique twisted equilibrium payoff. Hence, Bertrand's model is an example of a game of type I. But it is not a game of type C : for every $i \in N$, $\pi_i(c, 2c) = 0$ and $\pi_i(2c, 2c) > 0$.

Remark.

We assumed that firm i can choose any price $p_i \geq c$. If we lay down that firm i must choose an integer price (i.e. $p_i = c + k_i$ for some $k_i \in \mathbb{N}$), then Bertrand's model is a game of type III: (c, c) is a Nash and a twisted equilibrium, and $\frac{D(c+1)}{2}$ is a Nash

equilibrium payoff, but not a twisted equilibrium payoff. In particular, there are many Nash equilibrium payoffs.

9.2 ELECTORAL COMPETITION

N voters have to choose between two parties [11]. X denotes a finite set of alternatives. For each voter $k \in N$, we assume that there exists an order $\succeq_k$ over X . Each party chooses an alternative x_i , and receives a “payoff” $u_i(x_i, x_{-i})$ which represents the aim of the party i . We denote $n_i(x_i, x_{-i})$ the total number of voters for party i : $n_i(x_i, x_{-i}) = \#\{k \in N : x_i \succeq_k x_{-i}\}$. Laffond, Laslier and Le Breton assume that:

1. the parties have symmetric roles ($u_1(x_1, x_2) = u_2(x_2, x_1)$),
2. $u_1 + u_2 = 0$. Two types of payoffs are introduced:

- The majority game:

$$u_1(x_1, x_2) = \begin{cases} 1 & \text{if } n_1(x_1, x_2) > n_2(x_1, x_2) \\ 0 & \text{if } n_1(x_1, x_2) = n_2(x_1, x_2) \\ -1 & \text{otherwise} \end{cases}$$

- The plurality game: $u_1(x_1, x_2) = n_1(x_1, x_2) - n_2(x_1, x_2)$.

Laslier [12] generalized these games supposing that each party can take ambiguous positions: for example, during the electoral campaign, party i seems favorable sometimes to alternative x , sometimes to alternative x' in such a way that at the end of the campaign a proportion $p_i(x)$ of voters associate the party i with the alternative x . We assume that each party can control his campaign talk in such a way that it can generate a belief $p_i \in \Delta^X$. p_i is called platform of party i . Laslier assumes that the process of identification is independent:

- i) between the two parties; and
- ii) independent from the preferences².

DEFINITION 9.1 [12] *The plurality game with platform is the symmetric zero-sum game $(\Delta^X, \Delta^X, n_1 - n_2, n_2 - n_1)$.*

The majority game with platform is the symmetric zero-sum game $(\Delta^X, \Delta^X, \text{Sgn}(n_1 - n_2), -\text{Sgn}(n_1 - n_2))$.

The plurality game with platform is the same game as the mixed extension of the plurality game³. Moreover, Laslier [12] has shown that the plurality game with platform and the majority game with platform have the same equilibria set.

We assume now that the aim of each party is not to maximize the size of his majority but to maximize the size of its electorate. When $n_1 + n_2 = N$, these two hypothesis

2. i.e. For every profile of preferences $(\succeq_k)_{k \in N}$, for every $x_1, x_2 \in X$, the proportion of voters who associates party 1 to alternative x_1 and party 2 with x_2 is equal to $p_1(x_1) \cdot p_2(x_2)$.

3. But the majority game with platform is not the mixed extension of the majority game.

are equivalent *i.e.* $(\Delta^X, \Delta^X, n_1, n_2)$ and $(\Delta^X, \Delta^X, n_1 - n_2, n_2 - n_1)$ have the same equilibria. Assume now that $n_1 + n_2 \neq N$. For example, we suppose that $n_1 + n_2 = N - n_A$ where n_A is the number of abstainers⁴. The equivalence between $(\Delta^X, \Delta^X, n_1, n_2)$ and $(\Delta^X, \Delta^X, n_1 - n_2, n_2 - n_1)$ is no longer true:

EXAMPLE 9.1 $X = \{x, x'\}$. The percentage of voters for each party is given by the matrix:

$$\begin{array}{cc} & \begin{array}{cc} x & x' \end{array} \\ \begin{array}{c} x \\ x' \end{array} & \begin{pmatrix} .1, .1 & .1, .02 \\ .02, .1 & .2, .2 \end{pmatrix} \end{array}$$

(x', x') is an equilibrium of $(\Delta^X, \Delta^X, n_1, n_2)$ but not of $(\Delta^X, \Delta^X, n_1 - n_2, n_2 - n_1)$.

But if we assume that the game $(\Delta^X, \Delta^X, n_1, n_2)$ is of type *C*, then property 8.2 shows that indicate with a number remains true. To say that $(\Delta^X, \Delta^X, n_1, n_2)$ is a game of type *C* means that if party 1 has strictly more voters when it chooses platform p_1 rather than p'_1 against p_2 , then the electorate of party 2 may be stable, for example because the change from p_1 to p'_1 has attracted the sympathy of voters which would have abstained otherwise, but may not increase.

10 CONCLUDING REMARK

We studied classes of two-person non zero-sum games having properties satisfied by two-person zero-sum games such as uniqueness of Nash equilibrium payoff. These classes are defined by weakening the conflicting aspect of zero-sum games in two ways:

- By comparing couples of strategies according to comparison rules based on payoff functions (games of type *A* [8], *B* and *C* [9]).
- By comparing strategic behavior of the players with the help of the notion of twisted equilibrium (games of type I [1], II, III and IV).

We showed that all these games have a unique Nash equilibrium payoff, and that it is also the unique correlated equilibrium payoff of games of type *A*, *B* and *C*.

Definitions of games of type *A*, *B* and *C* are also available in the case of finitely many players [9]. As in the two-person case, they have a unique Nash equilibrium payoff [6, 9]. One could ask how to extend definitions of games of type I, II, III and IV to the n -person case. This is the subject of our future research.

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4. For example, $n_i(p_i, p_{-i}) = \#\{k \in N : p_i \geq_k p_{-i} \text{ and } k \text{ prefers } p_i \text{ to abstain}\}$.

BIBLIOGRAPHIE

- [1] AUMANN, R.J., "Almost strictly competitive games", *Journal of the SIAM* 9, 1961, p. 544-550.
- [2] AUMANN, R.J., "Subjectivity and correlation in randomized strategies", *Journal of the Mathematical Economics* 1, 1974, p. 67-96.
- [3] AUMANN, R.J., "Correlated equilibria as an expression of bayesian rationality", *Econometrica* 55, 1987, p. 67-96.
- [4] AUMANN, R.J. and MASCHLER, M., "Some thoughts on the minimax principle", *Management Science* 18, 1972, p. 54-63.
- [5] DE WOLF, O., *Jeux compétitifs et équilibres corrélés*, Mémoire de D.E.A., Université catholique de Louvain, 1997.
- [6] DE WOLF, O., *Optimal strategies in n-person unilaterally competitive games*, C.O.R.E. DP 9949, 1999.
- [7] FORGES, F., "An approach to communication equilibria", *Econometrica* 54, 1986, p. 1375-1385.
- [8] FRIEDMAN, J., "On characterizing equilibrium points in two-person strictly competitive games", *International Journal of Game Theory* 12, 1983, p. 245-247.
- [9] KATS, A. and THISSE, J.F., "Unilaterally competitive games", *International Journal of Game Theory* 21, 1992, p. 291-299.
- [10] KREPS, D., *A course in microeconomic theory*, Princeton University Press, 1990.
- [11] LAFFOND, G., LASLIER, J.-F. and LE BRETON, M., "Social choice mediators", *American Economic Review* 184, 1994, p. 448-453.
- [12] LASLIER, J.-F., "Interpretation of electoral mixed strategies", *Social Choice and Welfare* 17, 2000, p. 283-292.
- [13] LEININGER, W., THORLUND-PETERSEN, L. and WEIBULL, J., *A note on strictly competitive and zero-sum games*, Universität Bonn DP A-194, 1988.
- [14] MOULIN, H., "Cooperation in mixed strategies", *Mathematics of Operations Research* 1, 1976, p. 273-286.
- [15] MOULIN, H. and VIAL, J.-P., "Strategically zero-sum games: the class of games whose completely mixed equilibria cannot be improved upon", *International Journal of Game Theory* 7, 1976, p. 201-221.
- [16] ROSENTHAL, R., "Correlated equilibria in some classes of two-person games", *International Journal of Game Theory* 3, 1974, p. 119-128.
- [17] SORIN, S., "On repeated games with complete information", *Mathematics of Operations Research* 11, 1986, p. 147-160.
- [18] SORIN, S., "Communication, correlation and cooperation", *Cooperation: Game-Theoretic Approaches*, Springer-Verlag, 1997.

means *A is included in B*

means *C is not included in D*

Figure 1. Connections between the different classes