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On A. A. Markov's Attitude towards Brouwer's Intuitionism

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Résumé : Le présent article traite de l'attitude critique d'Andreï A. Markov à l'encontre de l'intuitionnisme de L.E.J. Brouwer telle qu'elle se manifeste dans ses notes à la fin de la traduction russe de l'*Intuitionnisme* de Heyting, publiée à Moscou en 1965. On considère que l'approche algorithmique de Markov s'est développée sous l'influence du style mathématique et des valeurs régnant au sein de l'Ecole mathématique de Saint-Petersbourg, qui est caractérisée par la primauté accordée aux applications et à la recherche de la rigueur et de solutions efficaces.

Abstract: The paper examines Andreï A. Markov's critical attitude towards L.E.J. Brouwer's intuitionism, as is expressed in his endnotes to the Russian translation of Heyting's *Intuitionism*, published in Moscow in 1965. It is argued that Markov's algorithmic approach was shaped under the influence of the mathematical style and values prevailing in the Petersburg mathematical school, which is characterized by the proclaimed primacy of applications and the search for rigor and effective solutions.

Dedicated to Vadim A. Yankov

1 Introduction

In 1956, Arend Heyting (1898-1980) published his *Intuitionism* [Heyting 1956], that could be characterized as the first compendium of the form of mathematics which was inaugurated by Luitzen E. J. Brouwer (1881-1966) in his dissertation of 1907. In order to help the reader, who may not necessarily think intuitionistically, understand the controversies between intuitionistic way of

FIGURE 1: Vadim A. Yankov at the Moscow M.V. Lomonosov University on 4th October 1991. Photo by Ioannis M. Vandoulakis.

thinking and other standpoints in the foundations of mathematics, Heyting adopted a dialogical form of exposition. Thus, the book is articulated in the form of dispute between the fictional persons “Class”, “Form”, “Int”, “Pragm”, and “Sign”, which represent classical mathematics, formalism, intuitionism, pragmatism and significicism, respectively. Nevertheless, it also has the feature of systematic exposition, because it is divided into eight chapters,¹ covering the fundamental parts of intuitionistic mathematics.

Nine years after the original English publication, in 1965, the book was translated into Russian and published in Moscow by Mir Publishers that was specialized in translation of scientific literature during the Soviet era. The translator was Vadim A. Yankov (1935-), one of Andreĭ A. Markov’s (1903-1979)² pupil. The later served as the editor. Andreĭ A. Markov is known as the founder of the Soviet school of constructive mathematics. On suggestion by Yankov, Markov added in endnotes a new person—“Con”—representing the constructivist, in Markov’s sense. “Con” expresses Markov’s viewpoint, as it was developed until that time and enjoys a privileged status: he has the advantage of ultimate criticism of all aforementioned representatives, without being liable to criticism. However, Markov focuses his criticism predominantly against Brouwer, whom he perceives as his principal antagonist, and essentially ignores the other representatives, except David Hilbert (1862-1943) whose program to “save” the “precious” mathematical results he characterizes as a pointless undertaking (“What to save and why?”).

1. 1. Disputation, 2. Arithmetic, 3. Spreads and species, 4. Algebra, 5. Plane point species, 6. Measure and integration, 7. Logic, 8. Controversial subjects.

2. Often referred to as “the junior” in the mathematical literature, to distinguish him from his father, Andreĭ A. Markov (1856-1922), who is known for his contributions in probability theory, mathematical analysis and number theory.

Markov's criticism is focused on certain principal points that we consider below. A disadvantage of putting Markov's exposition in the endnotes, that occupy 33 of the 200 pages of the Russian edition, is that Markov cannot provide a systematic exposition of his constructive mathematics, like Heyting's exposition of intuitionistic mathematics. In compensation, Markov provides essential references to works published in Russian up to that time that clarify his viewpoint on questions discussed in Heyting's exposition.

These endnotes are very important from an historical point of view, because they are the only written document that expresses Markov's attitude towards Brouwer's intuitionism. The text shows that Markov was very critical to Brouwer, at that time. Research in Markov's archive by Nikolai N. Nagornyĭ (1928-2007) has shown that at a later time Markov unexpectedly started striving to bring closer his standpoint to Brouwer's views [Markov & Nagornyĭ 1984, 14; 2010, xix]. Heyting took into account Markov's endnotes in the third edition of his book, in 1971. Thus, these endnotes serve also as a starting point of a historical "dialogue" developed between the mathematical schools of Brouwer's intuitionism and Markov's constructivism.

Accordingly, we focus in this paper on some central questions, on which Markov has explicitly expressed his views. These endnotes succeed in clarifying the principal differences between Markov's and Brouwer's viewpoints on the foundations of mathematics. Although many of his comments are mostly technical, his style remains lively expressive and revealing of Markov's different states of disposition: some comments are critical, or even ironic, others express Markov's satisfaction or disagreement with Brouwer's views.

2 Markov's principal objections against Brouwer's intuitionism

2.1 Different understanding of constructive objects

In the very beginning of chapter 1, Heyting lays stress on the fact that intuitionism is concerned with mental mathematical constructions, which require a different form of logic.

Markov agrees that mathematical constructions require a different form of logic. However, he disagrees with the view that the process of construction is extra-linguistic or *mental*. He understands it as *real*, like a process executable in a computer. Thus, by *constructive objects* Markov understands not mentally conceived objects, but concrete objects, like the letters of an alphabet, that is a (finite or infinite) collection of discernible signs.

Accordingly, Markov rejects Brouwer's assumption that certain objects of mathematics and mathematical operations are sufficiently *evident*, so that manipulation of these objects by such operations cannot lead to inconsistencies—

FIGURE 2: Andrei A. Markov at the Computing Center of the Russian Academy of Sciences. Photo donated by N. M. Nagornyĭ.

assumption that is grounded on Kant's concept of intuition (*Anschauung*)—and bypasses his view of mathematical propositions (formulae, equations, etc.) as mere images (*Bilder*) of free mathematical creation.

Consequently, in Markov's view, constructive mathematics study *constructive processes* and *constructive objects* generated by them; this needs a new form of logic—the constructive (in Markov's sense) mathematical logic.

2.2 On the concept of potential infinite

At the same passage, Markov objects to Heyting's view that mathematics from the very beginning “deal with the infinite”. Markov claims that the infinite is introduced in mathematics by abstraction (idealization). He distinguishes between the “unclear” (in Markov's view) *abstraction of the actual infinity*, which is used to introduce (unintuitable) complete infinite totalities, and the *abstraction of potential realizability*³ that abstracts away from any practical spatial, temporal or material limitations in our capacity of constructing (concrete or abstract) mathematical objects. This abstraction enables us to conduct reasoning on as lengthy constructive processes and as large constructive objects as required. Thereby, as constructive objects can be considered only those, which are not generated by abstractions more powerful than the abstraction of potential realizability.

Markov assumes a philosophical stand about abstractions in the late 1950s:

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3. This term was introduced by Markov during the second half of 1940's.

Abstractions are necessary in mathematics; however, they must not be devised for their own sake and lead where there is no return down to "earth". We should always remember to pass from abstract thinking to practice, as a necessary step of human cognition of objective reality. In case that the possibility of such a passage is turned out to be too doubtful, it is necessary to reconsider the abstractions applied and try to modify them. [Markov 1958, 315–316]

Proceeding in line with this thesis, he understands Brouwer's mental constructions as potentially realizable, since they have (practically) realizable material constructions as archetypes. In this way, Markov actually reinterprets Brouwer's idea of *potential infinite* in terms of his own concept of abstraction of potential realizability, in an attempt to "return down to earth".

2.3 Mathematical existence

Heyting's claim that in the study of mental mathematical constructions "to exist" must be synonymous with "to be constructed" could not pass unnoticed by Markov.

Markov notes that it is not necessary that any definition indicates an object, falling under this definition, because the construction of such an object can be an open problem; thus, we cannot claim that the object exists, unless we solve the problem. This is the case with definition II of natural number in Heyting's book [Heyting 1956, 2].

Hence, Markov identifies mathematical existence with potential realizability of a construction; however, this is not perceived as a process evolving in time (like in Brouwer's concept of creative subject or Kripke's scheme). An object exists whenever it can be indicated as a complete finite word (in an alphabet) or it is given by a pair (letter, algorithm) and it is known that the algorithm is applicable to the letter. In case that such a pair cannot be constructed or the applicability of the algorithm cannot be established, this does not mean that the object does not exist. An object does not exist only whenever the impossibility for the object to be constructed is proved (for instance, if the inapplicability of the corresponding algorithm is proved). In this case, the object under consideration does not exist eternally.

2.4 The question of classification of constructive mathematics

The question "under which sciences intuitionistic or constructive mathematics should be properly classified?" is examined in relation to the understanding of

constructive processes. In Heyting's view, mathematics studies certain functions of the human mind and therefore it is more akin to philosophy, history and the social sciences [Heyting 1956, 10].

Markov objects to this classification on the grounds that the human mind together with its mental constructions are parts of nature. Such mental constructions, as the construction of greater and greater natural numbers, have material archetypes in reality. On the other hand, mental constructions, such as complex algorithms, are initially conceived as mental constructions, but are implemented afterwards as computer programs. Therefore, mental constructions do not fall under social sciences.

Markov slightly develops this viewpoint in an unfinished manuscript written during the last months of his life that was posthumously published by N. Nagornyĭ in 1987. In this manuscript, Markov unreservedly classifies constructive mathematics under technological sciences,

because [constructive mathematics] investigate and supply instruments, applied in various spheres of human activity. In this respect, it is like engineering. [Markov 1987, 212]

Thus, Markov's view on constructive mathematics is a viewpoint of a specialist primarily interested in applications of mathematics.

2.5 The concept of number

In chapter 2, Heyting explains the concept of natural number and clarifies the difference of the intuitionistic concept from other ways of conceiving natural numbers. Markov does not perceive that there is any essential difference of his view with the intuitionistic understanding of natural number. Specifically, for Markov natural numbers are defined as words of the form $|$, $||$, $|||$, etc., over the alphabet $|$. The abstraction of potential realizability does not allow the formation of “infinite” words or the collection of “all” words over a given alphabet taken for completed entity.

However, Markov notes that Heyting's concept of sequence of rational numbers is unclear and possibly not constructive. He explains that in his view of constructive mathematics, rational numbers are understood as words of a certain type over the alphabet $\{ |, -/ \}$ (“ $-$ ” is the sign of minus, $/$ is the sign of fraction). A sequence of rational numbers is a (normal) algorithm that maps every natural number into a rational number.

2.6 On Church thesis

In connection with Brouwer's examples depending on unsolved problems, Markov develops a lengthy critical argument in defense of *Church thesis*, stated independently by Alonso Church (1903-1995), Emil Leon Post (1897-1954),

and Alan Mathison Turing, (1912-1954) in 1936 [Olszewski, Woleński *et al.* 2006, 7] and expressing the fact that certain refinements of the concept of algorithm (such as, for instance the concepts of recursive function, of λ -definable function, Turing machines, etc.) are adequate explications of the broad intuitive concept of algorithm.

In Markov's constructive mathematics, Church thesis assumes the form of the *principle of normalization of algorithms*, which states that every verbal algorithm in an alphabet V is equivalent with respect to V to some normal algorithm in V , or, concisely, every verbal algorithm is normalizable.

Your examples are very pleasant and subtle. Each of them is based on some problem unsolved at present. You are obviously convinced that as soon as the problem you use would be solved in one or another way (which may comfortably happen), you will immediately invent another example of the same kind, based on another unsolved problem.

Let us free ourselves to imagine that some genius mathematician invented a general method (an "algorithm") that enables us to solve any single mathematical problem, i.e., to give a correct answer—"yes" or "no"—to any mathematical question, requiring such an answer. Then you will not be able to invent *none* such problem and you would be apparently compelled to agree on everything with Mr. Class [the representative of classical mathematics]. You are possibly afraid of such a tragic perspective. Of course, you are aware that Church proved the undecidability of the decidable problem [Church 1936] and that there has been proved today several modest "massive" mathematical problems. However, all these results are based on the one or the other refinement of the concept of algorithm ("unified general method"), for instance on the concept of recursive function, and the assumption about the adequacy of this refinement, for instance, on Church Thesis asserting that "recursiveness" is equivalent to "calculability". It is clear that without digging into the concept of algorithm, no proof of impossibility of decidable algorithm may pass through. If you were not willing to accept Church Thesis or some version of it, then you would be compelled to agree that your divergence with Mr. Class depends on the state of our knowledge at the present time. All your anti-classical propositions should be then considered as *de facto* truths, not as *de jure* truths. Do you feel comfortable with that?

On the other hand, if you accept Church Thesis and modern theory of algorithms, then this would entail a substantial reform of your mathematical outlook and the transition from intuitionism to constructive understanding of mathematics. [Heyting 1956, 163–164]

Church Thesis is a point of fundamental divergence between intuitionism and Markov's constructive mathematics. Heyting considers this thesis in two of his subsequent papers [Heyting 1962, 1969], arguing against its adoption. In view of this criticism, Markov is obliged to give explanations about the possible applications of his own principle of normalization of algorithms and indicate explicitly wherever is used in [Markov & Nagorny 1984].

2.7 On the concept of infinitely proceeding sequence

In chapter 3, Heyting defines the most important concept in intuitionistic mathematics, that of the *spread* and the *fan theorem* is proved. Before the definition of the spread, Heyting introduces the concept of *infinitely proceeding sequences*, which is not defined by a definite law, but can be an object of ever-creating mental construction.

Markov severely attacks against this concept. He considers that the concept of infinitely proceeding sequence is not at all evident and, possibly, even non-constructive. It is not clear why Markov considers that the concept of infinitely proceeding sequence is non-constructive. A possible explanation might be the indeterminacy or eventual impossibility of practical realization of the acts of determination of the successive components of the sequence. This explanation is suggested by the following comment by Markov:

I cannot but feel sorry for the man, whom you are ready to force to do so many [acts of] “free choice” or “dice drops”. My understanding of infinitely proceeding sequence is more human, since algorithms can be executed easily by a computer. And what is most important is that my understanding is constructive. Because the concept of algorithm can be standardised, which makes possible the coding of an algorithm and its recording by “letters” in a fixed alphabet. In turn, algorithms themselves can become constructive objects. It is possible to apply other algorithms to them, which is very important in constructive analysis.

Your infinitely proceeding sequences are not constructive objects, and I cannot manage them. [Heyting 1956, 166]

Thus, Markov suggests a constructive, in his sense, reinterpretation of the concept of infinitely proceeding sequence in terms of *normal algorithm*⁴ [Markov 1954b] and rejects infinitely proceeding sequences as non-constructive objects. Thereby, existence of a mathematical object explicitly means for Markov its *algorithmic*, not mental, construction.

Accordingly, Markov reinterprets Heyting's definition of spread (given in terms of the spread-law and the complementary law), by substituting the

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4. Markov uses the word *algorifm* (with “f”) to denote “normal algorithm”, instead of the common Russian word *algoritm* (with “t”).

concept of law by the concept of (normal) algorithm. As a consequence of Markov's constructive understanding of spreads, the fan theorem is no longer true. The fan theorem is refuted by a counter-example, on the grounds of several theorems proved by I. D. Zaslavskii [Zaslavskii 1962].

2.8 Not all intuitionistic theorems are true in constructive mathematics and vice versa

Markov also attacks certain intuitionistic theorems that are not true in his own constructive mathematics, outlining a rather complicated picture: some intuitionistic theorems are refutable in Markov's constructive mathematics, whereas there are theorems in Markov's constructive mathematics that do not hold in intuitionistic mathematics.

Thus, the intuitionistic form of the Heine-Borel theorem holds true, but it is refutable by a counterexample in Markov's constructive mathematics [Zaslavskii 1962]. The theorem that a bounded function, which is defined almost everywhere is measurable,⁵ holds true in intuitionistic mathematics, but it is refutable by a counterexample in Markov's constructive mathematics [Zaslavskii & Ceitin 1962].

On the other hand, the proposition that every constructive function of a real variable is continuous everywhere in its domain of definition holds true in Markov's constructive mathematics [Markov 1958], but not in intuitionistic mathematics.

Nevertheless, Markov's notes do not offer an overall comparison between his own version of constructive mathematics and intuitionism. Certain fundamental divergences are not considered here. For instance, the so-called *Markov's principle of constructive selection* [Markov 1954a, 1956, 1962], according to which if a constructive process, given by some prescription, is not potentially infinite, then the process terminates. This principle is essential for the proof of certain theorems in Markov's mathematical analysis. However, it is not acceptable by the intuitionists, because of the *ad hoc* use of an indirect argument in it and remained controversial and insufficiently evident even among some Markovian constructivists [Kushner 1973, 45].

Relatively few notes are devoted to Logic [Heyting 1956, chapter 7], although Markov's constructive semantics has deep differences from the intuitionistic one. The semantics for Markov's constructive logic was a later development; it is based on the idea of hierarchy of (formal) languages [Markov 1974a,b,c,d,e,f,g]. Markov does not state either any new argument against the Law of Excluded Middle.

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5. Markov objects also to Heyting's definitions of *measure* and *measurable region* (in chapter 6) and suggests his own constructive reinterpretation of these concepts [Heyting 1956, 178].

3 On the reception of Brouwer's ideas in the Soviet Union

The reception of the ideas of intuitionism in the Soviet Union was extremely diverse. Soviet philosophers of mathematics, ardent champions of dialectical materialism, for instance, Vladimir Nikolaevich Molodshii (1906-1986), although they were not satisfied with the underlying Platonism of George Cantor's set-theoretic conception, they were also hostile to the ideas of "effectivists", i.e., Émile Borel (1871-1956), Henri Lebesgue (1875-1941), René-Louis Baire (1874-1932), which were close to some of Brouwer's ideas. The ideas of this group of French mathematicians were perceived as "subjective idealism" and variation of Ernst Mach's (1838-1916) philosophy of science [Molodshii 1938, 53, 78]. The hostility was strengthened by the fact that Nikolai Nikolaevich Luzin (1883-1950), who was viewed as an adherent to the ideas of the French "effectivists" was accused as "active counter-revolutionary" and persecuted [Molodshii 1938, 78-84]; [Demidov & Levshin 1999]. In general, Hilbert's formalism and Brouwer's intuitionism were considered as trends of idealistic philosophy [Molodshii 1938]. Even such a notable mathematician, as Aleksandr Jakovlevich Khinchin (1894-1959) does not escape from a subjective exposition of Brouwer's ideas [Khinchin 1926].

A decisive step towards demystification of Brouwer's intuitionism in the Soviet Union was done by Andrei Nikolaevich Kolmogorov (1903-1987) whose early works on intuitionistic logic [Kolmogorov 1925, 1932] contributed to its establishment as a mathematical discipline, rather than a philosophical doctrine. Kolmogorov's ideas exerted influence on certain historians and philosophers of mathematics, notably Adolf Pavlovich Yushkevich (1906-1993) and Sofya Aleksandrovna Yanovskaya (1896-1966). They knew each other from Odessa,⁶ and cooperated in propagating Hilbert's and Brouwer's ideas. In particular, Yushkevich translated in 1934 Herman Weyl's (1885-1955) book *On Philosophy of Mathematics* [Weyl 1934] with an introduction by Yanovskaya, that for long remained in the Soviet Union a primary source of information about the Western developments in philosophy of mathematics. Thus, Brouwer's ideas became initially known to the Soviet scholars through Weyl's perception of them. Two years later, Yushkevich translated Heyting's *Survey of Research on Foundations of Mathematics* [Heyting 1936], with an introduction by Kolmogorov. Moreover, Yanovskaya seems to have been influenced by Kolmogorov's criticism concerning the applicability of the Law of Excluded Middle [Yanovskaya 1936, 88].

Gradually, the anti-Platonic orientation of Brouwer's philosophy of mathematics started to be attractive to Soviet philosophers, because it was considered compliant with the principles of dialectical materialism. During the

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6. Sofya Yanovskaya was one of Yushkevich's teachers in a gymnasium when the Yushkevich family returned to Odessa after 1917.

time of the thaw, the severe criticism gave its place to efforts of releasing the mathematical content of Brouwer's intuitionism from his general philosophical viewpoint. In this line, Brouwer's concept of *intuition* was studied, particularly by the Soviet philosopher Valentin Ferdinandovich. Asmus (1894-1975) [Asmus 1963], who distinguished the concept of intuition as used in the context of mathematical problems and mathematical creative imagination from intuition as used in philosophical contexts, which is irrelevant to mathematics. During this period, Markov's constructive mathematics was flourished and perceived as an approach on foundations of mathematics, alternative to intuitionism.

Nevertheless, it has never acquired the status of a trend in Soviet philosophy of mathematics. This was partly caused by the fact that the mathematicians of Markov's school, including Markov himself, abstained from expressing publicly their philosophical views; they preferred to remain on the solid ground of mathematical proving activity. Moreover, the school included repressed and persecuted mathematicians, such as Nikolai M. Nagornyĭ and Vadim A. Yankov, mentioned in this paper. In particular, Vadim Yankov, the translator and initiator of the exposition of Markov's viewpoint in Heyting's book, was involved in the dissident movement, arrested in 1982 and sentenced to four years in prison and three years in exile. He was given amnesty and released in January 1987, and rehabilitated in October 30, 1991. Since then he works in the Russian State University for Humanities.

4 Markov in the context of opposition between the Moscow and Saint Petersburg schools

Markov's reaction to intuitionism goes beyond the horizon defined by Kolmogorov's interpretation of intuitionistic logic in 1932. His interest in foundations of mathematics was shaped not on the grounds of philosophical discussions, which were prevalent in the Moscow school of mathematics, even during the 1930s, in the circles of Ivan Ivanovich Zhegalkin (1869-1947) and A. N. Kolmogorov, but of his research background in applied mathematics and the theory of algorithms, developed during the time he was in Leningrad.

Traditionally, there was a rivalry between the Moscow and the Saint Petersburg Mathematical Schools. The mathematicians of the Moscow school were tolerant to various philosophical conceptions concerning mathematics and mathematical objects; this tendency goes back to D. F. Egorov (1869-1931), N. N. Luzin (1883-1950), P. A. Florensky (1882-1937), and others [Demidov 1993]. On the contrary, the mathematicians of Saint Petersburg school shared values that were primarily of positivist and Western orientation.

This discrepancy assumed the form of open conflicts throughout history, especially after Pafnuty L. Chebyshev's (1821-1894) death. A vivid expression of the opposition between the two schools is the alleged judgment of the Professor of Saint Petersburg Academy Vladimir Andreevich Steklov (1863-1926) on young Luzin's dissertation *Integral and Trigonometric Series*. After browsing the work of his Moscow colleague he asked:

where the formulas are here? This is not mathematics, but some philosophy! [Demidov 1999, 414]

Markov grew out of the intellectual environment of the Leningrad school. All his educational background and early carrier are connected with Leningrad. In 1935 he became a Doctor of Science in Leningrad University; the next year, he was nominated Professor at the same university. Since 1939 he has worked at the Leningrad Branch of the Steklov Mathematical Institute. While he was in Leningrad, his research was focused on general theory of dynamic systems, topology and measure theory; particularly, he studied algorithmic problems of topology, theory of computable invariants of binary relations, cryptography, etc. Since 1946, he has turned to the theory of algorithms and recursive functions that led him to the introduction of the concept of *normal algorithm*, independently from the foundational debates.

Moving to Moscow in 1955, Markov brings with him, the arsenal of his theory of algorithms as well as the traditional positivist dispositions of the intellectual environment of the Leningrad school. By his “algorithmic” approach, Markov tends to “free” intuitionism from its underpinning metaphysical assumptions. It would be rather superficial to interpret this fact as an impact of the governing ideology of dialectical materialism. Markov's attitude can be better explained as an impact of positivistic disposition and his earlier research on applied mathematics and algorithmic problems, during the period he spent in Leningrad (1933-1955).

Thus, his research orientation in Leningrad was shaped under the influence of the mathematical style and values prevailing in the Saint Petersburg Mathematical School, which is characterized by the proclaimed primacy of applications and the search for rigor and effective solutions [Demidov 1999, 415]. This spirit determines also his criticism of Brouwer's intuitionism and his tendency to “free” it from its underlying philosophical assumptions.

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