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Jan von Plato

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Logic as Calculus and Logic as Language: Too Suggestive to be Truthful?

Jan von Plato

University of Helsinki (Finland)

Résumé : Le présent article porte sur le rôle inférentiel des quantificateurs chez Frege, Peano et Russell. Nous abordons ici deux aspects qui caractérisent la logique mathématique à ses débuts : le progressif perfectionnement des principes de raisonnement à l'aide des quantificateurs d'une part, la prétendue impossibilité conceptuelle de poser des questions de type métathéorique d'autre part, telle qu'elle est incarnée par le célèbre dicton de Jean van Heijenoort sur « la logique comme calcul et la logique comme langage ».

Abstract: The paper focuses on the inferential role of quantifiers in Frege, Peano and Russell. Two aspects of the early years of mathematical logic are discussed: the gradual perfection of the principles of reasoning with quantifiers, and the presumed conceptual impossibility of posing metatheoretical questions, as embodied in Jean van Heijenoort's well-known dictum about "logic as calculus and logic as language."

1 Introduction

Modern logic is usually counted to begin with the times of George Boole, around 1850, with an initial phase of algebraic logic. It was gradually replaced by the logic of Frege and Peano, conceived in 1879 and 1889, and received through the mediation of Russell in the first decade of the 20th century. In this essay, I shall discuss two aspects of these pioneer years: the principles of reasoning with the quantifiers, and the awareness about metatheoretical questions. Jean van Heijenoort, in a widely read essay "Logic as calculus and logic as language" [van Heijenoort 1967*b*], claims that such questions were a conceptual impossibility for the early pioneers of modern logic; a suggestive thesis that turns out to be in great part a myth not supported by any close reading of the sources.

2 Quantifiers in Frege, Peano, and Russell

The alternatives for the formal treatment of foundational questions in mathematics at the end of the 19th century were—excepting Frege whose work was not understood by others—Schröder’s algebraic logic and Peano’s axiomatic systems, both lacking essential principles of reasoning with the quantifiers.

The very first appearance of generality and existence were in Aristotle’s syllogistic: *every A is B* and *some A is B*. These quantifiers were, though, not variable-binding operations and therefore did not play any clear role in ancient mathematics [see von Plato 2017, chap. 1].

Traditional modes of inference with existence and universality include that one can take instances of a universal and conclude existence from an instance. When mathematical proofs got formalised, these were written as axioms, with t a term that names a constant:

$$\text{I } \forall x A(x) \supset A(t) \quad \text{II } A(t) \supset \exists x A(x)$$

To infer the other way around, to generality in I, the traditional pattern has been:

III Assume a counterexample, $\exists x \neg A(x)$, and if this turns out impossible, conclude $\forall x A(x)$.

For this pattern to be useful, one needs to take instances of $\exists x \neg A(x)$, a step that was never made formally precise before 1920.

A principle that relates to II is:

IV Assume the contrary of $\exists x A(x)$ for all cases, namely $\forall x \neg A(x)$, take instances by axiom I, and if something impossible turns out, conclude $\exists x A(x)$.

Whereas in II existence is inferred from an instance, in IV it is proved *indirectly*.

Principles III and IV are based on the intuitive equivalences between $\neg \exists x \neg A(x)$ and $\forall x A(x)$, and $\neg \forall x \neg A(x)$ and $\exists x A(x)$. The former, in particular, has been very prominent, already endorsed by Aristotle (namely that generality means the lack of a counterexample).

No systematic theory of inferences with the quantifiers appeared before Frege discovered such in 1879, but it remained unrecognised for some 25 years. Peano, for example, had axiom II, and had even invented the symbol $\exists$, but expressed generality by free variables, typically the inductive step in arithmetic in which one proves for a property of numbers the implication $A(n) \supset A(n+1)$. If n is “arbitrary” and if $A(0)$ holds, the free-variable formula $A(x)$ can be inferred. Then any substitution instance can be taken, analogously to axiom I, but Peano recognised no general logical pattern here,

contrary to Frege, and his formal systems of mathematics remained severely incomplete. This feature was, in the end, responsible for the failure of the Peano school. Peano himself dismissed Frege's discovery by writing in 1895 in a reply to a letter of Frege's that his approach with just three instead of Frege's five basic signs "corresponds to a more profound analysis."

Frege explains in his *Begriffsschrift* [Frege 1879] the truth of a universal proposition by stating that each of its instances is true. Then comes the "illuminating observation" that one can infer to generality if one has proved an *arbitrary instance*, the latter meaning that a proof of $A(a)$ has been given for a "Latin letter." The condition is that nothing on which $A(a)$ depends, given as an antecedent B in an implication, must contain the "Latin letter." Frege expressed the universal quantifier by a "German letter" (fraktur) and a notch in the "judgment stroke." In modern notation, $B \supset \forall x A(x)$ can be inferred from $B \supset A(a)$ when a purely syntactic criterion on the condition B is fulfilled. Today this is seen as the greatest of logical discoveries:

Russell's 1903 book *The Principles of Mathematics* contains an appendix "The logical and arithmetical doctrines of Frege," with Frege clearly singled out as the one to explicitly recognise the difference between "any and every," and the need of rules for handling the latter:

He has a special symbol for assertion, and he is able to assert for all values of x a propositional function not stating an implication, which Peano's symbolism will not do. He also distinguishes, by the use of Latin and German letters, respectively, between any proposition of a certain propositional function and all such propositions. [Russell 1903, 519]

These were afterthoughts. No explicit logical notation is used in *The Principles of Mathematics* proper, but the treatment is based on Peano's work. Russell thinks he can get along with a single primitive notion in logic, what he called the "formal implication" rendered as " φx implies ψx for all values of x " [Russell 1903, 11]. Peano had used the notation $\varphi x \supset_x \psi x$ for such an implication with a free variable, typically an eigenvariable in an inductive step from x to its successor x' . That was the nature of Peano's arithmetic: there was no universal quantifier in his formalism. It is easy for us today to understand that this is the systematic reason for the failure of his attempt to formalize mathematics and for the demise of his school, but such insights come from hindsight: Russell, for example, tells in the preface to his book that he had seen Frege's *Grundgesetze der Arithmetik* [Frege 1893] but added that he "failed to grasp its importance or to understand its contents," the reason being "the great difficulty of his symbolism" [Russell 1903, xvi].

A somewhat neglected paper of 1906, "The theory of implication" [Russell 1906], is Russell's first contribution to the deductive machinery of logic. Published four years before the *Principia*, it shows clearly the origins of Russell's formal system of proof. In this work, he uses negation and implication

as primitives, with a handsome axiomatization as a result. As shown in my *Formal Machinery* [von Plato 2017, section 5.1], the structure of derivations in Russell is *identical* to Peano's, with two formal rules of inference: to take instances of axioms, and to apply the rule of detachment.

Van Heijenoort, who edited the book that contains the first English translation of the main part of Peano's 1889 work [Peano 1889], instead of figuring out what Peano's notation for derivations means, claims in his introduction that there is "a grave defect. The formulas are simply listed, not derived; and they could not be derived, because no rules of inference are given [...]" he does not have any rule that would play the role of the rule of detachment" [van Heijenoort 1967a, 84]. Had he not seen the forms $a \supset b$ and $a.a \supset b : \supset b$ in Peano's derivations, the typographical display of steps of axiom instances and implication eliminations with the conclusion b standing out at right, and the rigorous rule of combining the antecedent of each two-premiss derivation step from previously concluded formulas? Had he not seen the identical structure in Russell's 1906 article? Van Heijenoort's unfortunate assessment has undermined the view of Peano's contribution for a long time, when instead Peano's derivations are constructed purely formally, with a notation as explicit as one can desire, by the application of axiom instances and implication eliminations.

Russell in his 1906 article goes beyond Peano by introducing the notation $(x)C(x)$ for universal quantification, presumably the first such notation in place of Frege's notch in the assertion sign, alongside the Π_x notation in Schröder's algebraic logic. The notation for universal quantification allows to express the rule of generalization as:

If $C(y)$ is true whatever y may be, then $C(x)$ is true for all values of x . [Russell 1906, 195, slightly modified]

Russell's formal notation is $\vdash C(y) \supset \vdash (x)C(x)$, i.e., from the derivability of $C(y)$ "whatever y may be," $(x)C(x)$ can be inferred.

The universal quantifier makes its next appearance in Russell's famous 1908 paper on the theory of types. Its section II is titled "All and any" and contains:

The distinction between all and any is, therefore, necessary to deductive reasoning and occurs throughout in mathematics, though, so far as I know, its importance remained unnoticed until Frege pointed it out. [Russell 1908, 2289]

Mathematical reasoning proceeds through *any*:

In any chain of mathematical reasoning, the objects whose properties are being investigated are the arguments to *any* value of a propositional function. [*ibid.*, 227]

Still, reasoning with just free variables would not do, for bound variables are needed in definitions (Russell’s terminology for free and bound variables is “real” and “apparent”). Remarkably, his example is from mathematics proper:

We call $f(x)$ continuous for $x = a$ if, for every positive number $\sigma \dots$ there exists a positive number $\varepsilon \dots$ such that, for all values of δ which are numerically less than ε , the difference $f(a + \delta) - f(a)$ is numerically less than σ . [*ibid.*]

He goes on to explain that f appears in the definition in the *any*-mode, as an arbitrary function, and that σ, ε , and δ instead are just “apparent variables” without which the definition could not be made.

3 First-order logic

If we take the classical propositional calculus and add to it the rules and axioms for universality, a complete classical first-order theory of quantification emerges. An understanding of this matter was slow in coming: The study of Russell’s logic in Göttingen, from around 1917 on, led in some eight years to the first impeccable formulation of the axioms and rules of first-order logic. The first publication of all the axioms and rules, with universality and existence both taken as primitive notions, was in Hilbert and Ackermann’s *Grundzüge der theoretischen Logik* of 1928 [Hilbert & Ackermann 1928]—a book actually written by Paul Bernays—in which it is acknowledged that the axiomatisation of quantificational logic was found by Bernays.

Existence was a primitive notion in Hilbert-Ackermann for the somewhat casual reason that the book emphasised prenex normal form in which both quantifiers appear only as alternating strings at the head of formulas, followed by a propositional part as in the example $\forall x \exists y \forall z (A(x, y) \& A(y, z) \supset A(x, z))$. There is another, subtler reason not revealed by Hilbert-Ackermann, namely, the impossibility to define $\exists$ in terms of $\forall$ in intuitionistic logic. When first-order intuitionistic logic was definitively axiomatised by Heyting in 1930, he received a letter of congratulations from Bernays: The latter wrote that after Brouwer’s visit to Göttingen in 1925, he had figured out the axiomatics of intuitionistic logic so that classical logic results if the law of double negation is added. (The letter is found in Troelstra 1990.)

The rule of inference for existence is to instantiate $\exists x A(x)$ by a freshly chosen *arbitrary* variable y , in the form of an *assumption* $A(y)$, where arbitrariness has the same meaning as with universal generalisation. Whenever a consequence C of $A(y)$ and possible other assumptions is reached, the assumption $A(y)$ can be deleted if the *eigenvariable of existential instantiation* y has no occurrences in the other assumptions nor C . In typical cases, $\exists x A(x)$ is itself an existential assumption, and it replaces $A(y)$ by the step that is classified as “existence elimination” in terms of Gentzen’s natural deduction.

It is instructive to see, in the light of hindsight, how logicians failed in getting the principles of inference with the quantifiers right. One who got them right, but did not formalise the quantificational part, was Skolem in his long paper of 1920 [Skolem 1920]: When an existential *assumption* is put into use, that happens by taking an instance with “new letters.” In Skolem’s case, existence elimination was signalled by the use of Greek letters, only for this purpose.

In Kolmogorov’s 1925 paper on intuitionistic logic [Kolmogorov 1925], there is a clear awareness that the principles of inference with generality cannot be just axioms, but there has to be a rule that “cannot be expressed symbolically,” just as there has to be at least one rule of inference in axiomatic propositional logic.

Clamorous misses with the principles of quantificational logic include Rudolf Carnap’s little treatise *Abriss der Logistik* of 1929 [Carnap 1929], meant as a concise presentation of the logic of Russell’s *Principia*: The most central component of Russell’s logic is absent from Carnap’s presentation. Yet another famous logician who never saw the necessity of a rule of generalisation is Alfred Tarski: Again, one searches in vain for this rule in his path-breaking treatise on the concept of truth in formalised languages [Tarski 1935]. C. H. Langford’s 1930 review of the Hilbert-Ackermann book is yet another witness of how difficult it was to even entertain the idea of a formal system of rules for the quantifiers, a review full of nonsense but still published in the *Bulletin of the American Mathematical Society* [Langford 1930].¹ Last in this line comes Ludwig Wittgenstein who tried to do truth-tables even with quantified formulas.

Kurt Gödel had studied the *Principia* in the summer of 1928. Next he found the completeness problem, clearly spelt out in Hilbert-Ackermann [Hilbert & Ackermann 1928] that he also studied carefully. Gödel, clearly, would have no completeness theorem for first-order logic had he missed the crucial rule of generalisation. It is instructive to locate the exact point in which it is used in his 1930 proof [Gödel 1930]. The step is rather well hidden in the presentation that proceeds in terms of satisfiability. At one point, Gödel moves to provability of a free-variable formula, then universally quantified “by 3,” the number given for the rule of generalisation.

Not long after Gödel, young Gerhard Gentzen gave the definitive formulation of the axioms and rules for quantifiers, in his system of natural deduction. His approach has a most remarkable property, never seen before, that the principles of reasoning with each connective and quantifier are *pure*. By pure is here meant: formulated independently of the other connectives. In particular, the principles for universal and existential quantification are fixed once and for all, and whether a logic is classical or intuitionistic, say, is decided on another

1. This blunder was repaired by Barkley Rosser’s review of the second edition of 1937, with a closely identical title.

level. Previously it was thought that quantificational inferences would be the “dubitable” part of classical logic that could lead to contradictions.

A particular feature of natural deduction is the rule of existence elimination that has two premisses independent of each other: 1. The existential formula $\exists xA(x)$, either assumed or derived, and 2. A derivation of some consequence C from the assumption $A(y)$ in which y is the arbitrary eigenvariable, i.e., one that does not occur anywhere but in this subderivation, and not in C . These two independent premisses are displayed in a two-dimensional scheme:

$$\frac{\begin{array}{c} A(y) \\ \vdots \\ \exists xA(x) \quad C \end{array}}{C}$$

Even implication elimination and some other rules with independent premisses, say A and B in the introduction of a conjunction $A \& B$ are thus displayed, which opens the possibility to analyse the structure of derivations in first-order logic to the full for Gentzen, with well-known results about the normalization of derivations and the subformula property of such normal derivations [see von Plato 2008].

On a more general level, Gentzen’s systematisation of quantificational logic answers all questions about the meaning of quantifiers and the role of free-variable formulas, by dividing such questions into two: 1. The meaning when universality and existence are asserted, given through the conditions for their introduction. 2. The meaning when they are assumed, given through the elimination rules that show how such assumptions are put into use. Free-variable formulas are governed by the restrictions in universal introduction and existence elimination. The contrast with axiomatic logic is great, for there free-variable formulas are interpreted as equal in meaning to their universal closures.

4 “The universality of logic”

One of van Heijenoort’s famous theses about Frege, and by implication about Russell and to a lesser extent Peano, concerns “the universality of logic.” That notion is meant to illuminate an important aspect of Frege’s and Russell’s logic, as in the widely read essay “Logic as calculus and logic as language:”

Another important consequence of the universality of logic is that nothing can be, or has to be, said outside of the system. And, in fact, Frege never raises any metasytematic question (consistency, independence of axioms, completeness). [van Heijenoort 1967*b*, 13]

Van Heijenoort wrote such things under the authority of Gödel who in his paper on the completeness of predicate logic had stated that with a system such as the *Principia Mathematica*, the question of completeness “*at once* arises.”

Frege’s *Begriffsschrift* made little compromise in the direction of the reader. In the long preface to the *Grundgesetze* some fifteen years later, Frege was more forthcoming when explaining formulas in proofs:

Each of these formulas is a complete sentence together with all the conditions that are necessary for it to hold. This completeness, one that will not tolerate assumptions that could be added tacitly, seems to me to be indispensable for the rigorous carrying through of proofs. [Frege 1893, v–vi]

Completeness is used here in two senses, first the grammatical completeness of a sentence, then the complete display of the conditions under which a formula holds. This display is further explained as follows:

It cannot be required that all things be proved because that is impossible; but one can require that all sentences that are used without proof are expressly stated to be such, so that one can see clearly on what the whole edifice depends. One must, also, try to diminish the number of these ground laws, by proving all that is provable [*was beweisbar ist*]. Moreover, and here is where I go beyond Euclid, I require that all forms of inference and consequence that come to be used are listed in advance. [Frege 1893, v–vi]

The condition of mutual independence of the axioms is expressed by a notion of *provability*.

Frege identified the classical propositional calculus as defined through the primitive notions of conditional and negation, and the semantical criterion of correctness of formulas and inferences through two truth values formulated in terms of *admission* and *denial*, rather than simply truth and falsity, a step from an abstract notion of logical truth to semantical principles of reasoning.

Frege does not give any explicit axiom system, but the list of deductive dependences at the end of the *Begriffsschrift* identifies one. The question of its completeness seems to be an aspect Frege did not question but simply believed in, by his explanation of “the derivation of the more composite judgments from simpler ones” [Frege 1879, 25]:

In this way, one arrives at a small number of laws that contain, if one adds those contained in the rules, the content [*Inhalt*] of all the others, even if in an undeveloped way.

Consistency of at least the propositional axiom system is immediate, because Frege shows how to prove that his axioms must be “admitted” by semantical

criteria and that such admission is maintained by the only rule of inference, implication elimination.

Peano's "Sul concetto di numero" [Peano 1891], an article in his newly established *Rivista di Matematica*, relates his 1889 axiomatization of natural numbers to the axioms of Dedekind [Dedekind 1888]. The latter made him realise that the axioms can be taken in an abstract way, as follows:

These propositions express the necessary and sufficient conditions for the objects of a system to correspond univocally to the series of the N ; and they can be enunciated also as follows:

1. The name 1 is given to a particular object of the system.
2. Let an operation be defined for which there corresponds to every object a of the system another, $a+$, even that in the system.
3. And that two objects, the correspondents of which are equal, be equal.
4. The object called 1 shall not be the correspondent of any.
5. And finally that it be the class common to all classes s that contain the individual 1 and that, when they contain an individual, they also contain its correspondent.

It is easy to see that these conditions are independent. [Peano 1891, 93]

Browsing further in the *Rivista*, one finds in volume VI of 1899 Peano's notes on the *Formulario* project with the following passage:

The composition of my work of the year 1889 was still independent of the mentioned script of Dedekind; I had, before the printing, the moral proof of the independence of the primitive propositions from which I began, those with the substantial coincidence with the definitions of Dedekind. Later I succeeded in proving the independence. [Peano 1899, 85]

In the 1889 booklet on the foundations of geometry, we find a similar admission that the independence of the geometric axioms is a "moral certainty."

Peano's last exposition of his arithmetic was in the fifth edition of the *Formulario Mathematico* of 1908, written in his own invented language "Latino sine Flexione." On page 15, he explains:

We prove that a system of primitive propositions is mutually independent, in an absolute way, if we adduce, for each proposition, an interpretation of the system of primitive ideas that satisfies each primitive proposition except the one considered. [Peano 1908, 15]

Such proofs of independence of the Peano axioms are given on page 27 of the *Formulario*—so much for the presumed absence of metatheoretical questions in Peano.

With Russell, van Heijenoort takes an equally cavalier attitude as with Frege:

Questions about the system are as absent from *Principia mathematica* as they are from Frege's work. Semantic notions are unknown. [van Heijenoort 1967a, 14]

The notion of derivability that Russell expresses by the Fregean turnstile $\vdash$ is in van Heijenoort's reading a synonym for "is true." Russell is, indeed, confused about propositions, assertions, truth, and derivability, in a way Frege would never be.

Russell's 1906 article "The theory of implication" formulates a requirement of completeness in the following way:

Every deductive system must contain among its premisses as many of the properties of implication as are necessary to legitimate the ordinary procedure of deduction.

If it is our purpose to make all our assumptions explicit, and to effect the deduction of all our other propositions from these assumptions, it is obvious that the first assumptions we need are those that are required to make deduction possible. [Russell 1906, 159]

Russell takes it for granted that a complete system of deduction exists, but he is unable to express the matter in precise terms. A bit later, the formulation is:

Now in order that one proposition may be inferred from another, it is necessary that the two should have that relation which makes the one a consequence of the other. When a proposition q is a consequence of a proposition p , we say that p *implies* q . Thus deduction depends upon the relation of *implication*, and every deductive system must contain among its premisses as many of the properties of implication as are necessary to legitimate the ordinary procedure of deduction. [Russell 1906, 159]

Consequence is here clearly a notion outside the formal system of deduction. It is the notion of implication defined by the truth value semantics of classical propositional logic. The requirement that "all our other propositions" be derivable is a clear condition of completeness. As to the mutual independence of the axioms, Russell writes:

In the present article, certain propositions concerning implication will be stated as premisses, and it will be shown that they are sufficient for all common forms of inference. It will not be shown

that they are all necessary, and it is probable that the number of them might be diminished. [Russell 1906, 159]

Here it is presumed that some purported axioms could be in fact derivable theorems. Russell summarises the discussion by:

All that is affirmed concerning the premisses is (1) that they are true, (2) that they are sufficient for the theory of deduction, (3) that I do not know how to diminish their number. But with regard to (2), there must always be some element of doubt, since it is hard to be sure that one never uses some principle unconsciously. The habit of being rigidly guided by formal symbolic rules is a safeguard against unconscious assumptions; but even this safeguard is not always adequate. [Russell 1906, 159–160]

Van Heijenoort's thesis that truth and provability in Russell are the same can be hardly maintained, for the reason that the unproved axioms must be true.

Russell's discussion covers the problem of consistency in propositional logic: by (1), the axioms are true, and the only rule of inference is one Russell sometimes formulated as: "inference consists in the dropping of true antecedents in implications," namely, if we have that p and $p \supset q$ are true, then even q must be true. Nothing more is needed for a consistency proof of an axiomatic system of classical propositional logic.

Later in his 1985 article on Herbrand, van Heijenoort writes:

The only question of completeness that may arise is, to use an expression of Herbrand's, an *experimental* question. As many theorems as possible are derived in the system. Can we exhaust the intuitive modes of reasoning actually used in science? [van Heijenoort 1985, 14]

Let's grant to van Heijenoort this point, but the ones about consistency and independence don't hold water, they are unsustainable. A counterexample to consistency would be a derivable formula that is not true, say $A \& \neg A$ or $0 \neq 1$. A counterexample to independence is given by a purported axiom that turns out to be derivable, the latter a purely syntactic notion. A direct method for proofs of independence is the one of Peano in which an abstract calculus gets different interpretations. A counterexample to completeness is a formula that is true but not derivable.

Metatheoretical notions such as consistency, independence, and completeness are meaningful enough when it is understood what would show such properties not to hold. Indeed, Paul Bernays found out in his *Habilitationsschrift* of 1918 that one of Russell's axioms in the *Principia* turns out to be derivable from the rest of the axioms, a result published in Bernays [Bernays 1926]. Who can deny, then, that a syntactic method for proofs of underivability in axiomatic logic was an impossibility in the times of Frege and

Russell? Had someone shown to Frege or Russell an implication that is valid by the criterion of two truth values, but underivable from the axioms listed in the *Begriffsschrift* or *Principia Mathematica*, these two authors would have declared their axiomatisation *incomplete*, something van Heijenoort would have to regard a conceptual impossibility. Frege, Peano, and Russell would have naturally taken incompleteness as a failure that can be repaired by the addition of some principle that had remained hidden. They all clearly believed that deductive arguments can be perfected.

Van Heijenoort's pet idea of the "universality of logic" gives a suggestive and incisive view of the early phase of mathematical logic, but it is mostly a fantasy when confronted with the sources.

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