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Jan von Plato

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Gödel's Reading of Peano's *Arithmetices Principia*

Jan von Plato

University of Helsinki (Finland)

Résumé: Pour préparer son article sur la logique mathématique de Russell (1944), Gödel a lu attentivement les *Arithmetices Principia* de Peano. Les six pages résumant l'œuvre péanienne qu'il a écrites en employant la sténographie de Gabelsberger contiennent une analyse remarquable de la structure formelle des preuves de Peano, en contradiction manifeste avec l'opinion commune selon laquelle le traité de Peano ne contiendrait aucun mécanisme déductif formel.

Abstract: In preparation for his article on Russell's mathematical logic (1944), Gödel read carefully Peano's *Arithmetices Principia*. His six pages of summary in the Gabelsberger shorthand contain a remarkable analysis of the formal structure of Peano's proofs which is diametrically opposed to the common view that Peano's treatise contained no formal deductive machinery.

1 Introduction

In early 1943, Gödel left behind an incredibly intensive period of formal work, the main aims of which had been to prove the independence of the axiom of choice and of the continuum hypothesis in set theory on the one hand, the consistency of analysis on the other, with possibly the cardinality of the continuum decided—his conjecture was that it is $\aleph_2$. All of this can be seen through a study of his 16 mathematical *Arbeitshefte*: *Heft* 4 was begun after he had arrived to Princeton for good in late March 1940, *Heft* 15, eleven notebooks later, has at page 57 the date 14.X.42, and *Heft* 16 is the last one, with notes for his planned article “Russell's mathematical logic” [Gödel 1944] from page 9 on. Most notebooks run toward a hundred pages in length.

There is a notebook of Gödel's from the early 1930s, called the *Altes Excerptenheft*, that contains summaries of his reading of works for the writing of his part of the planned joint book with Arend Heyting, the *Mathematische Grundlagenforschung* [Heyting 1934]. He was unable to finish in time, mainly because of his incredible meticulousness, but his texts for two chapters for the book have survived [cf. von Plato 2021a].

From page 95 on, the *Heft* of excerpts for the book is written in a different style, and it even mentions the planned Russell article by which this part is from the early 1940s. The summaries are mainly about the work of Russell and Frege. Peano is not among the sources covered, and even the earlier parts of the *Heft* contain just the *Formulaire* project of Peano [Peano 1895], and readings from Peano's journal *Rivista di Matematica*. There are no signs that he would have studied the *Arithmetices Principia* directly [Peano 1889].

Gödel wanted, I believe, to be well prepared for the task of writing the article on Russell's logic, and to take a deep look at the work of Peano who so much influenced Russell, he ordered a copy of the *Arithmetices Principia* from a library in Chicago in early 1943. There is a summary of his reading of Peano, with a beautiful first page in which he tried to reproduce the appearance of the title page of Peano's little treatise. His six-page summary is among a large collection of excerpts written on loose papers at different times, some clearly from the 1930s, others later, in reel 36 of the Gödel microfilm collection, frames 416–421, document number 050135.

One sorry little corner in the historiography of logic and foundations concerns Peano's formalization of the rules of logical inference. In Jean van Heijenoort's widely studied *From Frege to Gödel* [van Heijenoort 1967], the claim is that there is "a grave defect. The formulas are simply listed, not derived; and they could not be derived, because no rules of inference are given [...] he does not have any rule that would play the role of the rule of detachment" [van Heijenoort 1967, 84], [cf. also von Plato 2021b, this journal]. It is an assessment that can at best be taken as a sign of total blindness. Gödel had from early on seen that Peano has precise rules of inference that are, however, not made explicit. His 1932 summary of Peano's 1895 version of the *Formulaire des Mathématiques* is found in one of his *Excerptenhefte* where he lists Peano's propositional axioms, then writes:

Rules: implication and substitution of equals (not formulated but used).¹

Gödel's summary of the *Arithmetices Principia* leaves no space for doubt about the formal character of proofs in Peano; this will be evident from the text itself to follow. It is clearly at the center of his comments.

1. Found on frame 515 left part, reel 20 of the Gödel microfilm collection, written in a mixture of German and Gabelsberger shorthand. Transcription in German is: Regeln: Implikation und Einsetzung für gleiche (nicht formuliert aber angewendet).

Peano's treatise is written in Latin, a language in which Gödel was a great expert. The text that follows maintains the Latin, but I have transcribed and translated into English Gödel's German remarks that are written in the Gabelsberger shorthand. Peano's treatise is readily available online; the translation in [van Heijenoort 1967] can be of help with the Latin passages.

Gödel's summary turns into a detailed commentary on Peano's system of proof in the middle of his page 5.

2 Gödel's summary of the *Arithmetices Principia*²

1.

Arithmetices Principia
Novo Methodo Exposita

A

Ioseph Peano

in R. Academia militari professore

Analysin infinitorum in R. Taurinensi Athenaeo docente.

LABOR ET HONOR³

Augustae Taurinorum

Ediderunt Fratres Bocca

Romae

Florentiae

Via del Corso, 216–217

Via Cerretani, 8

1889

This is an exact transcription of the title page of the book that I had borrowed from the University of Chicago, 18. Jan – 4 Feb. 1943, number: QA 142.P35

2.

Sign. Tabula p. VI

Praefatio P III – V bad Latin: Hic difficultas,

arithmeticae applicationes [instead of ad] processus -i

2. Kurt Gödel Papers, Box 10a, Folder 39, item accessions 050135, on deposit with the Manuscripts Division, Department of Rare Books and Special Collections, Princeton University Library. Used with permission of Institute for Advanced Study. Unpublished Copyright Institute for Advanced Study. All rights reserved.

3. [Design of a circular stamp of the publishing house.]

caput scripti = the main thing, namely the precise formalization, beyond that, some theorems formulated unclearly

Citations:	Boole	Camb. & Dublin Math J.	1848
	Pierce	Americ. J.	III 15
			VII 180
	Mc Coll	Proc. Lond. Math. IX 9 X 16	1878
	Schröder	<i>Lehrbuch der Arithmetik und Algebra</i>	1873,
		<i>Der Operationskreis des Logikkalküls</i>	1877
	Jevons	<i>Princ. of Sci.</i>	1883

Logicae notationes p. VI –XVI

1. Signum $\supset$ significat “deducitur”
 $\subset$ est consequentia
2. Defined in general: $\neg aRb$ the same as $a \neg Rb$
in particular, $a \neg =_x \Lambda$ means: there exists a
Similarly $aR \cup Sb =_{Df} aRb \vee aSb$,
for example $< \cup = aRSb =_{Df} aRb . aSb$
For example, $a \supset \neg = b$ means $a \supset b . \neg(a = b)$

3.

3. classis sive entium aggregatio

$a \supset b$ means a is a part of b

4. Classes with one element are identified with individuals, i.e.:

If $k \subseteq s$, then:

$k \varepsilon s \equiv k$ is a unit class $\equiv (x, y)(x, y \varepsilon k \supset x = y . k \neq \Lambda)$

5. $[x \varepsilon]a$ means: solutiones vel radices conditionis a

Similarly $[(x, y) \varepsilon]a =$ the class of pairs (x, y) . The pair appears as a basic concept.

6. $\exists R x$ means $R^{\text{“}}x \quad \exists = \text{qui (quae)}$

In particular, $\exists < y \quad \exists D x$, the divisor of x

It follows that $x \varepsilon \exists \alpha y$ (the same denoted by $[\varepsilon <]y$ and then generally).⁴

Especially also: $\exists \varepsilon \alpha = \alpha$

7. $(x', y' \dots)[x, y \dots] \alpha =_{Df} \text{Subst}(\alpha^x_{x'} \alpha^y_{y'} \dots)$

8. $\varphi \varepsilon F^{\text{“}}s$. φ is a function over s (that is denoted by putting it ahead).⁵

Therefore $a + . \varepsilon . F^{\text{“}}N, + a . \varepsilon . N^{\text{“}}F^6$

9. $[\varphi]$ denotes the inverse. In case of ambiguity, the class is associated)

Square brackets in the above usage mean an inversion, namely

1. $[x \varepsilon](x \varepsilon a) = a, x \varepsilon ([x \varepsilon] \alpha) = \alpha$

4.

2. Let α be a formula that contains the letter x and φ the sign for the function thus defined. Then we have $\alpha = \varphi x$, therefore $\varphi = \alpha[x]$ and $\varphi x' = \alpha[x]x'$. Analogously for $\alpha = x\varphi$

4. That is probably an error in printing and should be: $[\varepsilon] < x$.

5. That the function F is defined for x , is written as $F(x) = F(x)$.

6. [The second comma should be inverted as in Peano, p. XIII.]

3. From $dF(x) = f(x)dx$ follows symbolically $F(x) = [d]f(x)dx$
 therefore $F(x) = \int f(x)dx$

10. Functions applied on classes are defined in the known way.

11. The powers of a “functionis postsignum” φ are written as follows: φn

Therefore, if $+$ means the operation $+1$, then $a + b$ means automatically the correct one.

Arithmetices Principia p. 1–20

There are nine axioms, namely five axioms of equality, reflexivity, symmetry, transitivity, invariance for N and for $+$, and further $1 \in N, N+1 \subseteq N, x+1 \neq 1$ for $x \in N$, complete induction, but the last axiom of equality contains $a+1 = b+1 . \Rightarrow a = b$

Example of a theorem:⁷ $\exists D(a, b) = \exists DM \exists D(a, b)$, i.e., the divisors of a and b are the same as the divisors of the Max of the divisors of a and b .

5.

§ 1 On numbers and addition

§ 2 Subtraction

§ 3 Maxima and minima (of sets of numbers) M, W in relation to sets

§ 4 Multiplication

§ 5 Powers

§ 6 Division, $x \mathcal{D} y = x$ is a divisor of y , analogously $y \mathcal{I} x$

§ 7 Various number-theoretic theorems without proof. $x \pi y$ (relatively prime)

§ 8 Rational numbers (here also $\frac{p}{1} =_{Dfp} p$)

§ 9 Irrational numbers (as “upper limit of arbitrary sets of rational numbers”) Add & Mult defined

§ 10 Sets of real numbers, especially treated are the operations, inside, outside, limit and different relations between these. Remark: Everywhere, only *positive* numbers are defined.

Abbreviation of proofs: No distinction is made between “because” and “and,” i.e., the theorems cited are always written as conjunctions of conditions. Each proof has the form (in which the theorems cited are left out):

7. [[Peano’s theorem 36, p. 11.]]

$$P : \supset . Q . \supset . R . \supset . S . \supset . T \text{ etc } . \supset . Y$$

and the theorem proved reads then $P : \supset . Q \supset Y$.⁸ P and Q , are, then arbitrary, and all of the previous theorems hold as conditions from R on [so they can be used in the proof]. There stand next to the P, Q, R the theorems cited in a conjunction [or obtained through substitution

6.

of their specializations] that are required for the derivation of the next theorem.⁹

Direct use is very often made of the substitution of equals for equals [on the basis of preceding theorems], in particular also λ -conversion, the distributivity of $x \varepsilon$, etc. Theses, hypotheses, numbers of preceding theorems (possibly with the indication of some substitution) are taken to be simply abbreviations for the formulas in question [these can also contain free variables]. A proof is, then, a chain of trivial implications that are constituted so that:

1-) There occur in the conditions for the proof a set of theorems earlier proved that can be simply put aside.

2-) Then all the members can be put aside so that only the first (or the first two) and the last remain. [Bound variables are often denoted the same as free ones, $x \varepsilon [x3]$.

The implications proved one after the other are simply the initial part of the entire chain, where the preceding implications separate more strongly than those that follow. This resembles much more the logic of assumptions than Russell.

A definition is, after Peano, a proposition of the form $V \supset . x = a$ in which x is a sign (or a combination of signs) for which no meaning had been given so far and a one for which a meaning had been already given.

With the definition of the rational numbers it is stated: $\frac{p}{q}$ est novum ens.

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8. P, Q can also be absent in theorems without or with a condition.

9. I.e., one obtains through a reference to this condition plain trivial implications, and one can strike over afterwards in the conditions that which has already been proved.

Bibliography

- GÖDEL, Kurt [1944], Russell's mathematical logic, in: *The Philosophy of Bertrand Russell*, edited by P. Schilpp, La Salle: Open Court Publishing, 123–153.
- HEYTING, Arend [1934], *Mathematische Grundlagenforschung: Intuitionismus, Beweistheorie*, Berlin; Heidelberg: Springer, doi: 10.1007/978-3-642-65617-0.
- PEANO, Giuseppe [1889], *Arithmetices Principia Nova Methodo Exposita*, Turin: Fratelli Bocca, partial English translation in [van Heijenoort 1967].
- [1895], *Formulaire de Mathématiques*, vol. I, Turin: Fratelli Bocca.
- VON PLATO, Jan [2021a], *Chapters from Gödel's Unfinished Book on Foundational Research in Mathematics*, s. n.
- [2021b], Logic as calculus and logic as language: too suggestive to be truthful?, *Philosophia Scientiæ*, 25(1), 35–47.
- VAN HEIJENOORT, Jean (ed.) [1967], *From Frege to Gödel, a Source Book in Mathematical Logic, 1879–1931*, Cambridge: Harvard University Press.