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Peano and the Debate on Infinitesimals

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Résumé : Le principal objectif de cet article consiste à mettre en évidence la thèse de Peano visant à rejeter les infinitésimaux. Dans un premier temps, nous nous concentrons brièvement sur le contexte culturel dans lequel sont apparues et se sont développées les considérations de Peano. Ensuite, nous examinons l'article de Peano de 1892.

Abstract: The main aim of this paper is to put Peano's opinion about the unacceptability of the actual infinitesimal notion into evidence. First we briefly focus on the cultural environment where Peano's considerations originated and developed. Then we examine Peano's article of 1892, "Dimostrazione dell'impossibilità di segmenti infinitesimi costanti" [Peano 1892].

1 Introduction¹

Between the 19th and 20th centuries the possibility of theoretically accepting, or not, *actual* infinitesimals was much debated.² Mathematicians who were interested in the foundations of mathematics often had different opinions. At first we think it is useful to outline the distinction between *actual* infinite and *potential* infinite, according to the classical tradition. The *potential* infinite is conceived as something to which it is always possible to add a certain quantity, while the *actual* infinite is the possibility of instantly imagining a collection, a whole that has no end.³ So, for instance, in Euclidean geometry, a half-straight line is generated by applying the first and second Euclidean postulates. For Aristotle the actual infinity cannot be accepted,

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1. This work is supported by INDAM group GNSAGA.
2. This debate takes place in the context of the rigorous foundation of the analysis [i.e., see Lolli 2004, 82–86].
3. For other general information [see Bottazzini 2018].

and this conception prevailed until the nineteenth century.⁴ Likewise, we can state concerning the infinitesimals. A potentially very small magnitude can be generated through the principle according to which given a magnitude there is always a smaller one (see Archimedes' postulate). On that subject, Paul du Bois-Reymond wrote:

The infinitely small is a mathematical quantity and has all its properties in common with the finite. [...] A belief in the infinitely small does not triumph easily. [...] A majority of educated people will admit an infinite in space and time, and not just an "unboundedly large". But they will only believe in the infinitely small with difficulty, despite the fact that the infinitely small has the same right to existence as the infinitely large. [du Bois-Reymond 1877, 152]; [English trans. in En. Wikipedia]

Once the infinite number is accepted as an infinity, this number does not stay in \mathbf{R} . The infinitesimal can be seen as its reciprocal. Infinitesimals and infinities naturally occur in any description of the infinity. E.g., a potential infinitesimal is $1/n_{|n=\infty}$. The inverse is an infinity: $1/(1/n) = n_{|n=\infty} = \infty$.

We will define below the notion of the actual infinitesimal.

To understand the notion of infinitesimal we must take into account the concept of real numbers. In fact real numbers are based on the Dedekind axiom and it is well known that:

Archimedes Post. + Cantor Ax. \Leftrightarrow Dedekind Ax. [see Benedetti 1937, 28–36]

But Archimedes' Postulate⁵ is not compatible with the concept of the actual infinitesimal. Hence, if one considers the Dedekind axiom then we are at odds with the acceptability of (actual) infinitesimals.

4. About the actual infinite, Desargues had written: "So every straight line is intended to be stretched indefinitely both from one side and the other [...]" [see Taton 1981, 99].

5. Let us recall Archimedes' Postulate (axiom):

Let a and b be two magnitudes, if for example $a < b$ then $\exists n \in \mathbf{N}$ such that $na > b$

The Dedekind axiom (1872) says:

IF a segment AB of a straight line is divided into two parts in such a way that

- each point x of segment AB belongs to one of the two parts
- the point A belongs to the first part and B to the second
- any point of the first part precedes any point of the second in the order established in AB ,

THEN there exists a point M of the segment AB (which may belong to one or the second part), such that each point of AB which precedes M belongs to the first part, and each point of AB which follows M belongs to the second part according to the assigned subdivision.

From Dedekind's axiom one can derive Archimedes' Postulate.

In this article, first we present the Cantorian attempt to show the unacceptability of the infinitesimal notion. Then we highlight the debate on non-Archimedean concepts in Italy. Finally, we analyze Peano’s position.⁶

2 Cantor and the unacceptability of the infinitesimals

Cantor examines the unacceptability of the actual infinitesimals. We will refer to a letter written by Cantor to Benno Kerry, dated 4 February 1887 [see Ehrlich 2006, 29–34]. As we know, Cantor conceives the actual infinite which plays a crucial role in his theory of transfinite numbers. Without this approach, he could not have constructed this theory. It may therefore seem strange that he does not accept the actual infinitesimals. At first he proposes the unacceptability of $1/\omega$, ω being the first transfinite ordinal. Following [Ehrlich 2006], and with our modifications, we expose the claimed (but not right) Cantorian proof. One considers a set Z of linear magnitudes (i.e., the extremes of straight segments of finite length as linear numbers), which we denote ζ_1, ζ_2, \dots . These numbers must satisfy the following axioms:

Ax. 1 : The set of ζ_1, ζ_2, \dots is a commutative semi-group for the addition, where the operation $\nu \times \zeta$, with $\nu \in \mathbf{N}$, is possible.

Ax. 2 : $\zeta_1 + \zeta_2 + \zeta_3 + \dots = s \in Z$.

If $1/\omega$ exists, it is an infinitesimal because it is an inverse of an infinite (ω). We will have the magnitude $\zeta = 1/\omega$ and that is:

$$\zeta \times \omega = 1. \tag{1}$$

If $\zeta_1 = \zeta_2 = \dots = \zeta_n = \dots = 1/\omega$ we could write (1) as:

$$\zeta_1 + \zeta_2 + \dots + \zeta_n + \dots = 1 \text{ (the sum on the left has } \omega \text{ addends)}. \tag{2}$$

The Cantor axiom (1872) says:

IF two classes of a straight line segments are such that

- no segment of the first class is greater than a segment of the second
- if a small segment σ , as small as one wants is prefixed, then there is a segment of the first class and one of the second class whose difference is less than σ .

THEN a segment exists which is neither less than any segment of the first class nor greater than any of the second.

It is possible to give a postulate which is equivalent to Cantor’s axiom. According to Veronese this postulate consists in the following assertion:

“Ip. VIII: If a segment, whose extremities always vary in opposite directions becomes indefinitely small, it contains a point out of the range of variability of its extremes” [see Veronese 1889, 150].

6. For further information see *infra* [Benci & Freguglia 2019, 2016], [Ehrlich 2006], [Borga, Freguglia *et al.* 1985] and [Gemignani 1993].

At this point, Cantor uses the following property:

If $s \in \mathbb{Z}$, with $s < 1$ then, in virtue of (2), $\exists n \in \mathbf{N}$ such that

$$\zeta_1 + \zeta_2 + \dots + \zeta_n > s. \quad (3)$$

Hence for equation (3), let $s = 3/4$, then we will have:

$$\zeta_1 + \zeta_2 + \dots + \zeta_n > 3/4. \quad (4)$$

Because $\zeta_1 = \zeta_2 = \dots = \zeta_n = \dots = 1/\omega$ then (4) can be worth another $\zeta_i n$ -tuple, so:

$$\zeta_{n+1} + \zeta_{n+2} + \dots + \zeta_{2n} > 3/4. \quad (5)$$

Adding (4) and (5) we achieve:

$$\zeta_1 + \zeta_2 + \dots + \zeta_n + \zeta_{n+1} + \zeta_{n+2} + \dots + \zeta_{2n} > 3/4 + 3/4 = 3/2 > 1. \quad (6)$$

Which is in contradiction with (2). Therefore if we consider $1/\omega$ we arrive at a contradiction. Actually Cantor uses the following axiom (see equation (3)):

[**Ax. 3**]: If $\zeta_1 + \zeta_2 + \dots + \zeta_n + \dots = s$, then $\forall s' < s$ and $\exists n \in \mathbf{N}$ such that

$$\zeta_1 + \zeta_2 + \dots + \zeta_n > s'. \quad (7)$$

But one can see that this axiom [Ax.3] is a variant of Archimedes' Postulate. Therefore one remains within Archimedean mathematics where it is impossible to give the definition of actual infinitesimals (see equation (9)). So this Cantorian proof is not tenable.

Cantor develops this theme and he reaches an attempt to prove the unacceptability in general of infinitesimals, independently from the case $1/\omega$, which is however an important case. A first draft of this general demonstration is presented by Cantor in a letter to Karl Weirstrass dated May 16, 1887 [see Ehrlich 2006, 41]. These Cantorian analyses were accounted for by some Italian mathematicians.

3 The debate in Italy

A very interesting debate concerning the infinitesimals took place in Italy in the pages of the *Rivista di Matematica*, whose director was Giuseppe Peano from 1891 onward. We find contributions by Giulio Vivanti [Vivanti 1891a,b], by Rodolfo Bettazzi [Bettazzi 1891, 1892] and by Peano himself [Peano 1892]. Peano had asked for an invitation to the discussion also for Giuseppe Veronese, but this last declined the invitation. However, Veronese wrote an article about Peano's proof in *Rendiconti del Circolo Matematico di Palermo* in 1892 [Veronese 1892].

Giulio Vivanti in his article of 1891 wrote:

It seems that the idea of the proof of Cantor is this. Asserting that ζ [infinitesimal] is a segment, it is equivalent to admitting that if we successively arrange a sufficiently large series of segments, all equal to ζ , upon a straight line, we shall of necessity eventually cover the assigned finite segment in its entirety; next Cantor states (and here there is a gap in his explanation) that if this is not possible by means of a finite series of segments, it is impossible by means of an infinite series as well, however extended the series might be.⁷ [Vivanti 1891a, 138]

But—according to Vivanti—in the case of infinite series (a case which we cannot exclude), if the segment is covered then it is assimilable to a *linear continuous* which is compliant with the Dedekind axiom. Therefore the infinitesimals cannot be accepted. Vivanti wrote another article [see Vivanti 1891b] in the same journal and in the same year. Rodolfo Bettazzi in his article “Osservazioni sopra l’articolo del Dr. G. Vivanti sull’infinitesimo attuale”, also published in 1891 in *Rivista di matematica*, wrote:

It is preferable to derive the definition of actual infinitesimal from the words of the author [Vivanti] [...] “when repeated any finite number of times, it (the infinitesimal) never constitutes [...] any finite determined quantity” than from the same author’s words [...] which are not well defined [...] according to which the infinitesimal would be obtained by means of division into infinitely many equal parts, since the word *repeat* is understood in the ordinary manner of multiplication.⁸ [Bettazzi 1891, 175]

Bettazzi claims that admitting (see (9) below) $n\alpha < \beta \forall n \in \mathbf{N}$ (where α is an infinitesimal with respect to β , and β is a finite magnitude) one never reaches the finite determined quantity β . But is it possible to arrive at β by means of n as *transfinite number*? For the *nature* of the linear magnitudes (segments), which Cantor even considers, “with a sufficiently large number of magnitudes α one can reach or exceed β ”. Therefore the infinitesimals could

7. See English trans. in [Ehrlich 2006, 78]. : “Il concetto della dimostrazione di Cantor sembra essere questo. Dire che ζ è un segmento equivale ad ammettere che, disponendo successivamente sopra una retta una serie abbastanza grande di segmenti tutti eguali a ζ , si debba di necessità arrivare a coprire per intero un segmento finito assegnato; ora Cantor stabilisce (e qui v’ha una lacuna nella sua esposizione) che, se ciò non è possibile mediante una serie finita di segmenti, non lo è neppure mediante una serie infinita, comunque estesa essa sia” [Vivanti 1891a, 138].

8. See English trans. in [Ehrlich 2006, 86]. “La definizione dell’infinitesimo attuale, meglio che dalle parole dell’autore [Vivanti] [...] per le quali l’infinitesimo si otterrebbe dalla divisione in infinite parti uguali, non ben definita, si ha dalle altre dello stesso autore, che fanno seguito a quelle citate: ‘[...] esso (l’infinitesimo), ripetuto un numero finito qualsiasi di volte, non forma giammai [...] una quantità finita determinata qualunque’ purché si prenda la parola *ripetere* nell’ordinario significato della moltiplicazione” [Bettazzi 1891, 175].

not be accepted (see Peano's proof). It remains to be established whether the transfinite numbers exhaust the "sufficiently large infinite numbers" (as requested above). But that—as Bettazzi says—is not proven. Bettazzi wrote in 1892 another article "Sull'infinitesimo attuale" in the same journal [see Bettazzi 1892].

The debate based on Veronese's ideas had an international readership which then led Hilbert to contemplate in a logically correct way non-Archimedean geometry. Veronese had also had controversial exchanges with mathematicians such as W. Killing (1895), Cantor himself [see Veronese 1896], L. Schoenflies [see Veronese 1897, 1898] and H. Poincaré (1904)⁹ to mention just a few. Briefly, Veronese's structure \mathbf{V} of numbers (according to our reconstruction) is the following:

- \mathbf{V} is an ordered field
- \mathbf{R} (real numbers) $\subset \mathbf{V}$
- (*) Exist $\alpha \in \mathbf{V}$ such that $\alpha > n, \forall n \in \mathbf{N}$

α is an *infinite* number, but it is not unique; in fact:

$\alpha + 1, 5\alpha, \alpha^2$, etc., are also infinite numbers.

If $x \in \mathbf{V}$ and for every $k \in \mathbf{N}$ we have: $|x| < 1/k$ then x is called *infinitesimal*.

Indeed actually Veronese proposes the replacement of Archimedes' postulate with its negation, that is, with the fact that there are magnitudes that cannot be compared [Veronese 1891]. Explicitly, Veronese gives the following principles.

Principle III. In \mathbf{R} there is no minimal interval (magnitude) if the zero is excluded.¹⁰

Principle IV. If an interval (XX') whose extremities always vary in opposite directions becomes indefinitely small, it always contains an element Y of \mathbf{V} distinct from X and X' .¹¹

This Principle IV establishes that in \mathbf{V} an infinitesimal (interval) is different from zero. If (XX') is considered as a set of points, this principle affirms $(XX') \neq \emptyset$.

Otto Stolz read [Veronese 1891] and it seems that he shares Veronese's ideas, even if he prefers the Cantor-Dedekind approach [see Stolz 1891, 16].

9. Poincaré in his review of Hilbert's *Grundlagen* calls into question Veronese who answers in [Veronese 1905], see also [Veronese 1909].

10. [Veronese 1889, 610], English trans. in [Ehrlich 2006]: "Princ. III. Nel sistema \mathbf{R} non vi è un intervallo (grandezza) minimo se si esclude lo zero."

11. [Veronese 1889, 612], (English trans. in [Ehrlich 2006]): "Princ. IV. Se l'intervallo (XX') i cui estremi sono sempre variabili in verso opposto diventa indefinitamente piccolo, esso contiene sempre un elemento Y di Σ distinto da X e X' " A comparison between this Principle and Veronese's Ip. VIII is interesting, see footnote 5.

So it is possible, through (*) to define the notion of *actual* infinitesimal (see (9) below). Tullio Levi-Civita in [Levi-Civita 1892-1893], based on Veronese’s teaching, introduces the concept of *monosemii* and A. Bindoni, another disciple of Veronese, shows in 1902 [see Bindoni 1902] Hilbert’s geometrical field is contained in \mathbf{V} .

4 Peano’s “proof”

Peano published an article in 1892 in *Rivista di matematica* entitled “Dimostrazione dell’impossibilità di segmenti infinitesimi costanti” [Peano 1892]. He examines the problem of the unacceptability of actual infinitesimal segments. He begins by this premise:

This problem [that is, on the acceptability or not of the actual infinitesimals], debated between Dr. Vivanti and Dr. Bettazzi on the *Rivista di matematica*, is very interesting especially in this period where on the hypotheses of their existence theories and printed volumes exist. Cantor replied negatively to hypotheses of their existence, but the proof that this illustrious mathematician gave is so concise that it was judged incomplete. The aim of this article is to develop this proof.¹² [Peano 1892, 59]

Our aim is to analyze this proof, regardless of Ehrlich. Peano considers a half-straight line of origin o , as a set P of points. It seems clear that for Peano an *ended segment* is an open interval on the half-line, so:

$$op \equiv \{x : o < x < p \text{ and } p \in P\}$$

o is called *origin* and p *ending*,

while a *segment* is an ended segment u that verifies the following properties:

- u is a proper subset of the half-straight line,
- every point y between o and a point $x \in u$ is also a point of u ,
- vice versa: every point $y \in u$ is between o and some other point x of u .

The set of ended segments is denoted by S , and that of segments by s . It is possible to establish the sum of ended segments and the multiple of an ended segment and when one ended segment is lower than another: “Let u and v be two segments, we say that v is less than u , or that u is greater than v , if v is a

12. “Questa questione, dibattutasi tra i dott. Vivanti e Bettazzi sulla *Rivista di matematica*, è assai interessante tanto più che negli ultimi tempi sull’ipotesi della loro esistenza si sono fatte teorie e stampati dei volumi. Ad essa rispose negativamente il Cantor; ma la dimostrazione che questo illustre matematico ne diede è così concisa che fu giudicata incompleta. Scopo della presente nota si è di sviluppare questa dimostrazione” [Peano 1892, 59].

segment ended at a point in u , that is the ending of v is a point of u ". Every ended segment is a segment, but not vice versa.¹³

∞u , which is called by Peano *multiple of infinite order* of u , denotes the set of points which stay on some of segments $u, 2u, 3u$, etc. or on their *upper limit*. Peano shows that ∞u is a segment but not an ended segment. At first he affirms that an ended segment u is such that when it is added to itself, it becomes a double segment that exceeds u itself, that is :

$$2u > u. \quad (8)$$

Of course the (8) can be iterated.

Then he proposes the well-known actual infinitesimal definition, so:

$$\begin{aligned} &\text{The segment } u \text{ is an infinitesimal related to the segment } v, \\ &\text{and we will write } u \in \{v|\infty\}, \text{ if every multiple of } u \text{ is lower than } v. \end{aligned} \quad (9)$$

So if u is an infinitesimal relatively to v , ∞u , being the upper limit of the multiples of an infinitesimal, is less than v , i.e., $\infty u < v$. In fact, ∞u cannot exceed v because it is still a multiple of u and all multiples of u , for the (9), must be less than v , even if ∞u is the upper limit of all multiples. Besides, Peano explicitly assimilates ∞ to \aleph_0 ,¹⁴ therefore the following operations make sense under the assumption that u is an infinitesimal.

$$u, v \in S, u \in \{v|\infty\} \Rightarrow (\infty + 1)u = \infty u. \quad (10)$$

$$u, v \in S, u \in \{v|\infty\} \Rightarrow 2\infty u = \infty u. \quad (11)$$

If u is an infinitesimal in comparison to v , we have a class ∞u that is a segment contained in v . Therefore if we add ∞u to the segment u , we obtain the segment $(\infty + 1)u$ and by adding u again we have $(\infty + 2)u$, etc. We can also add ∞u with itself and we obtain $2\infty u$. So in general, we can have all multiples of ∞u ; we can multiply ∞u by ∞ and we can have $\infty^2 u$, and so on.

But all these various segments obtained by multiplying u by the transfinite numbers of Cantor $[\aleph_0 \text{ as } \infty]$ are always equal to $[\infty u]$.¹⁵ [Peano 1892, 61]

13. Otherwise we would be in the context of Dedekind's axiom.

14. In the sense that it has the same arithmetic behavior. In fact i.e.: $\infty + 1 = 1 + \infty = \infty$ as $\aleph_0 + 1 = 1 + \aleph_0 = \aleph_0$; $2\infty = \infty$ as $2\aleph_0 = \aleph_0 \cdot 2 = \aleph_0$, etc. But on the contrary $\omega + 1 \neq 1 + \omega$, $2\omega = \omega$ and $\omega 2 \neq \omega$. Hence, Peano (for ∞) refers to Cantorian transfinite cardinal.

15. Peano says: "Risulta che, se u è un infinitesimo rispetto a v , la classe ∞u è un segmento contenuto in v . In conseguenza possiamo aggiungere ad ∞u il segmento u , ottenendo il segmento $(\infty + 1)u$, a cui aggiungendo u otteniamo $(\infty + 2)u$, ecc. Possiamo sommare ∞u con se stesso ottenendo così $2\infty u$, ed in generale possiamo formare tutti i multipli di ∞u ; possiamo moltiplicare ∞u per ∞ ed ottenere $\infty^2 u$, e così via". "Ma tutti questi varii segmenti, che si ottengono moltiplicando u per i numeri transfiniti di Cantor sono uguali tra loro".

In short, we have:

$$\begin{aligned} \forall M \text{ transfinite number: } Mu &= \infty u \\ \forall n \in \mathbf{N} : nu &< v \\ \infty u &< v. \end{aligned} \tag{12}$$

That is: $\forall M$ transfinite number: $Mu = \infty u < v$ and in particular:

$$2\infty u = \infty u < v. \tag{13}$$

In this way Peano shows ∞u (with u infinitesimal) is not an ended segment. Indeed—according to Peano’s opinion—equation (13) is in contradiction with the “nature” of the ended segments and their calculus, i.e., with equation (8). Thus, it is impossible to obtain the infinitesimal “as an element of finite magnitude”. But in fact equations (8) and (13) cannot be compared because (8) is in an Archimedean context and (13) is not. Hence, Peano’s proof is not tenable.

If equation (9) is satisfied, then u is an infinitesimal. But if u is a finite segment (that is, if infinitesimal segments do not exist—as Peano says) then ∞u represents the infinite half-straight line with origin o . As we said, Veronese wrote a brief article that appeared in 1892 on *Rendiconti del Circolo Matematico di Palermo* [see Veronese 1892] entitled “Osservazioni sopra una dimostrazione contro il segmento infinitesimo attuale”. This referred to Peano’s proof, arguing this same conclusion. Afterward he repeats his proposal given in [Veronese 1891].¹⁶

5 Some conclusions

Despite the fact that history does not deal with ifs and buts, we must observe the discussion on aversion to the infinitesimals, which could have been more opportunely based on well-argued explicit positions, instead of on untenable proofs which in fact need, in various forms, Archimedes’ postulate. Indeed, the proofs of the actual infinitesimals that we have examined in a more or less explicit way use Archimedes’ postulate. These proofs are unacceptable, even with recourse to the proto-physical concept of “nature” of segments and the related calculus. From a foundational point of view, Hilbert in [Hilbert 1899, in particular see §12 chap. II] analyzes the possibility of non-Archimedean geometry. But, in general, from a point of view pertaining to content, many analyses and proposals of theories in which one considers infinitesimals are still lacking sufficient rigor, except for the Levi-Civita proposal [Levi-Civita 1892-1893] and research on non-Archimedean fields. Among the explicit opinions

16. Among other of Veronese’s remarks we see that he interprets, unlike us, the symbol ∞ (used by Peano) with ω .

against the actual infinite and infinitesimals, we report those of Poincaré and of Russell. The first, wrote: “The actual infinity does not exist: the Cantorians have forgotten it and they have obtained the antinomies” [see Poincaré 1906, 316–317] and Russell affirms: “It follows that the infinitesimals, in order to explain the continuity must be considered unnecessary, erroneous and self-contradictory [...] we have shown that differential and integral calculus do not need infinitesimals” [see Russell 1948, 480, 510].

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