Philosophia Scientiæ

Travaux d'histoire et de philosophie des sciences

25-1 | 2021

The Peano School: Logic, Epistemology and Didactics

Peano's Reception in the USA. Wilson's Review of Russell's Principles

Gabriele Lolli

Electronic version

URL: <https://journals.openedition.org/philosophiascientiae/2843>

DOI: 10.4000/philosophiascientiae.2843

ISSN: 1775-4283

Publisher

Éditions Kimé

Printed version

Date of publication: 25 February 2021

Number of pages: 49-67

ISBN: 978-2-38072-000-6

ISSN: 1281-2463

Electronic reference

Gabriele Lolli, "Peano's Reception in the USA. Wilson's Review of Russell's Principles", *Philosophia Scientiæ* [Online], 25-1 | 2021, Online since 01 March 2021, connection on 04 November 2021. URL: <http://journals.openedition.org/philosophiascientiae/2843> ; DOI: <https://doi.org/10.4000/philosophiascientiae.2843>

Tous droits réservés

Peano's Reception in the USA. Wilson's Review of Russell's Principles

Gabriele Lolli

Scuola Normale Superiore di Pisa (Italy)

Résumé : Dans une recension des *Principles* de Russell datant de 1904, Edwin B. Wilson accorde une attention particulière aux travaux de Peano et de ses collaborateurs. Son but était de mieux les faire connaître aux USA, où leurs œuvres « étaient malheureusement peu diffusées et n'étaient en outre pas spécialement appréciées ». La reconnaissance dont Peano bénéficiait aux yeux de Russell est amplifiée par Wilson, ce dernier estimant que la logique de Peano est bien plus qu'un nouvel « outil logique », tout en considérant le logicien italien comme un précurseur du logicisme. Wilson s'oppose ainsi au jugement dépréciatif dont Peano faisait l'objet de la part de Poincaré. Dans le domaine de la géométrie en particulier, Wilson revendique diverses avancées accomplies par l'école de Peano par rapport à Hilbert en ce qui concerne la philosophie de la méthode axiomatique.

Abstract: In a review of Russell's *Principles* from 1904, Edwin B. Wilson pays great attention to Peano's work and that of his collaborators. His purpose was to make this work known in the USA where it "unfortunately is very little known and still less appreciated". Wilson expands Russell's well-known acknowledgement of Peano's influence on his own development, seeing in Peano's logic more than a new "mathematical tool", describing Peano as a kind of proto-logician, and defending him from Poincaré's criticisms. Especially in geometry, he vindicates several priority issues for Peano's school with respect to Hilbert in the philosophy of the axiomatic method.

1 Introduction

In 1904 Edwin Bidwell Wilson (1879-1964) wrote a review in the *Bulletin of AMS* of two books by Bertrand Russell (1872-1970), *The Principles of*

Mathematics [Russell 1903] and a French translation of the older *An Essay on the Foundations of Geometry* [Russell 1897, 1901a]. Wilson's review of these books [Wilson 1904] is interesting per se, as a description of the natural or naïve motivation of a logicistic view of mathematics, and we will briefly also dwell on this part. However one has the impression that the review had been primarily conceived as an opportunity to talk about Peano and his school, and to describe his work as a prelude to Russell's. In fact, while we do not know, we doubt that at that period any of Peano's paper had been translated into English and thus made known to the American scientific public.

In [Wilson 1904, 74, fn.*], Wilson refers readers to another review by Couturat of the same 1903 book by Russell which is mainly dedicated to Cantor and set theory: "So large is the work of Russell, that Couturat's review and our own supplement rather than overlap one another". However Ivor Grattan-Guinness (1941-2014) glosses both reviews together noting that [Couturat 1904] "concentrated on Cantor [...] having written elsewhere on the Peanists", while Wilson "dwelt on the Peanists, whose work [according to Wilson] 'is very little known and still less appreciated' in the USA (p. 76), referring to their four lectures at Paris [1900 International Congress of Philosophy]" [Grattan-Guinness 2000, 330]. So there seems to be an agreement with Grattan-Guinness in seeing [Wilson 1904] as focused on the Peanists.

Wilson had begun his career in mathematics with a Ph. D. at Yale in 1901 and the same year wrote a book on *Vector analysis* based on Josiah Willard Gibbs' (1839-1903) lectures. He then studied in Paris in 1902-1903, and for a while was interested in the foundations of geometry (he criticized "so-called" Hilbert's foundations in 1903). Next he began teaching at Yale and later at MIT in the department of physics. He was inspired by Gibbs to work in mathematical physics, first in mechanics and the theory of relativity, then in aeronautics and aerodynamics and later in statistics with applications in many fields. He wrote the first American advanced calculus text. His most demanding logical contribution, apart from a few reviews, is a discussion of categoricity, which we shall come back to later.

His time in Paris clearly brought him into contact with the lively discussions on foundations that were beginning in Europe. While he was in Paris, he possibly met Louis Couturat (1868-1914) who was devoting himself to promoting Russell's work. Wilson begins his review by explaining what is "the problem of the ultimate foundation of mathematics": throughout history,

[...] what has been accepted [in pure mathematics] as sure and accurate in one generation has frequently required fundamental revision in the next. Euclid and his pupils could doubtless have complained of the lack of rigour and logical precision in his predecessors just as forcibly as some modern pupils of Weierstrass

berate their scientific ancestors and companions.
[Wilson 1904, 74¹]

He mentions Euler, Cauchy, Laplace, and Dirichlet, and some of their inaccuracies or errors, which nonetheless did not hinder their work.

We notice that the advance toward our present rigour has been made step by step by great men who, however, were no greater—one might almost say no more careful—than their fellows working in apparent unconsciousness of the impending trouble and perhaps even incredulous at first as to its reality. When will this revision stop? And whereunto will it finally lead? This is the problem of the ultimate foundation of mathematics. (75)

This is a problem that cannot be easily disposed of by ignoring it:

the delicacy of the question is such that even the greatest mathematicians and philosophers of to-day have made what seem to be substantial slips of judgement and have shown on occasion an astounding ignorance of the essence of the problem they were discussing. (75)

At times this could have been caused by the failings of individual intuition in dealing with matters that are still unsettled, “but all too frequently it has been the result of a wholly unpardonable disregard of the work already accomplished by others” (75).

After this criticism of his fellow-mathematicians, Wilson enters the subject by quoting the first section of the first chapter of *The Principles*, containing Russell’s famous definition of pure mathematics as the class of all propositions, with variables, of the form “ p implies q ”, with a few qualifications (the logical constants allowed in p and q being: implication, to be an element of, such that, the notion of relation, and truth, plus “such further notions as may be involved in the general notion of propositions of the above form” (75)).²

2 Peano & Co.’s encomium

Russell’s was “probably the first attempt to give a complete definition of mathematics solely in terms of the laws of thought and the other necessary paraphernalia of the thinking mind”; it was made possible by “two things: first, the more careful discrimination of what *pure* mathematics is; second, the extraordinary development of logic since Boole removed it from the trammels of medieval scholasticism” (76). But

1. Hereinafter, page numbers in the text, (n), without further reference, indicate the source [Wilson 1904].

2. Russell’s presentation of symbolic logic as in the second chapter of the book was still in its infancy.

He to whom the presently highly developed state of the foundations of mathematics is chiefly due is Peano—one whose work unfortunately is very little known and still less appreciated in this country. (76)

Although Leibniz achieved a lot and his work in recent years has been made known by L. Couturat,³ George Boole (1815-1964) freed us from Aristotelianism, and C. S. Peirce (1839-1914) and Ernst Schröder (1841-1902) carried the technique of logic much farther,⁴

[this notwithstanding] they had never accomplished that intimate formal relation between logic and all mathematics which was the necessary precursor to a yet more intimate philosophic relation and which has been brought about by Peano aided by a large school of pupils and fellow-workers. The advance has been made largely by introducing into symbolic logic such a simplification of notation as to relieve it of its unwieldiness and to allow its development into a powerful instrument without which one can hardly hope to get the best results in the treacherous though treasure-laden fields of the foundations of mathematics. (76–77)

Wilson is however aware that not everyone concurs with his view and that Peano even had his detractors. Henri Poincaré (1854-1912) in particular (in his review of Hilbert's *Grundlagen der Geometrie*) “spurns this [Peano's] *pasigraphy*, characterizing it as disastrous in teaching, hurtful to mental development, and deadening for investigators, nipping their originality in the bud” (77).⁵ Even accepting the first statements, to which Wilson will return in the conclusions,

we had best be cautious in accepting such sweeping statements as the last, even from so great an authority—especially in view of the fact that, equipped with this *pasigraphy*, the Italian investigators, Peano and his pupil Pieri,[†] with some rights of priority, had given a more fundamental *logical*[‡] treatment of the subject on which Poincaré was writing than is to be found in the work he was praising so highly [the *Grundlagen*]. (77–78)

Footnote[†] (77) is simply a reference to [Pieri 1898] and [Pieri 1899]. Footnote[‡] (77) instead is a long note in which Wilson disputes some of

3. Wilson here quotes [Couturat 1901].

4. According to Wilson the importance of the logic of relations was emphasized by Peirce in 1880-1884, but only now begins to show its utility. He does not mention here, but later in the review, (80, fn.*), that the first exercise Russell did to master Peano's machinery was to treat the theory of relations symbolically, in [Russell 1901 b]. Wilson excuses Peano and his followers for neglecting the importance of this subject because they were “so busy [...] with other important questions”.

5. Wilson is quoting (77), from [Poincaré 1903, 5].

Poincaré's judgements. His own assessments in this footnote amount to a vindication of several priority issues for Peano's school.

Since Poincaré praised the "long step in advance" made by David Hilbert (1862-1943) in the philosophy of mathematics by regarding "his geometric elements as *mere things*", Wilson reminds us that Peano had already taken the same stance [Peano 1889] and that Giovanni Vailati (1863-1909) had expressed the same idea in [Vailati 1891] and [Vailati 1892]. He also notes that by 1897 Peano and his students had gone further by envisaging the postulate that points are classes, with a twofold advance: first of all the necessity of a postulate, secondly the use of the term "class", which permitted a further reduction of mathematical reasoning to logic.

As for attributing the idea of the independence of the axioms to Hilbert, in 1894 Peano had already stated the problem and given proofs of independence for certain axioms with simple systems of elements. "By 1899 the idea and method were both five years old at least" (77, fn.[‡]).

Of course, in Hilbert's work "there still remains [...] matter enough for the amplest praise":

The archimedean axiom, the theorems of Pascal and Desargues, the analysis of segments and areas, and a host of things are treated either for the first time or in a new way, and with consummate skill. We should say that it was in the techniques rather than in the philosophy of geometry that Hilbert created an epoch. (77, fn.[‡])

What Wilson is implying with this strong assertion is that the philosophy of the axiomatic method was already well established and widespread. This was in fact the subject of Vailati's two papers and he could have mentioned others geometricians, such as Moritz Pasch (1843-1930) for example.

Turning to arithmetic and algebra, Wilson is on slippery ground either because of his lesser competence, or for his propensity to take the claims of Alessandro Padoa (1868-1937) and Cesare Burali-Forti (1861-1931) at face value. Indeed he probably only read these in the summaries of their Paris 1900 papers without studying them, otherwise he would have been doubtful about or mystified by Padoa's alleged feat.

In his communication at the Congresses,⁶ Padoa had explained very precisely how to analyse the formal structure of a deductive theory. He had in a sense defined and set out the canon of the axiomatic method which Wilson largely draws from in section 4 of the review dedicated to the fundamental concepts of the axiomatic method. One of the principles was that

6. Padoa repeated his communication at both congresses, preceding it at the philosophy congress with a "logical introduction to any deductive theory". An English translation of this introduction is included in [van Heijenoort 1967, 118-123].

Pour **démontrer** la compatibilité d'un système de postulats, *il faut trouver* une interprétation des symboles non définis, qui vérifie simultanément tous les postulats. [Padoa 1903, 87]

On that occasion, Padoa had also addressed the problem of consistency posed by Hilbert in 1899 for the number system (in a conference that became [Hilbert 1900b]) and inserted as problem n. 2 in his list of problems, [Hilbert 1900a]. Padoa was probably disappointed by the silence that had greeted his contention when he claimed explicitly in 1903 that Hilbert's problem n. 2 was completely solved, and that actually "Le problème n. 2 [de M. Hilbert] n'était qu'une causerie" [Padoa 1903, 86].

What he had done was to define an interpretation saying that "**entier** signifie nombre entier relatif" and "**suc** x signifie $1 + x$ ". This is not very different from Poincaré's assertion (p. 8 of his review) that we know that axioms are non-contradictory "since geometry exists", to which Wilson implicitly takes issue (78, fn.[‡]).

Wilson however accepts that "a solution [of problem n. 2] has long since been proposed in the article here referred to" (78, fn.*), the reference being [Padoa 1903]. He admits nevertheless in a cryptic way that "There are those, however, who hold that Padoa has gone so far as to overshoot the mark" (78, fn.*). In the same footnote, Wilson regrets having overlooked the fact that a solution to the consistency problem along the lines proposed by Hilbert "seems logically impossible". What Hilbert asked for in 1900 was direct proof of the consistency of the axioms of arithmetic through "an appropriate modification of known methods of proof", applying only "the known inference methods of the theory of irrational numbers" [Hilbert 1900a, §41]. However Wilson acknowledges that "Hilbert has again taken up the matter much more searchingly than in 1900" (78, fn.*). He was also informed that Hilbert had proposed a new approach at the Heidelberg congress in August 1904 and regretted that he had not seen the text.

It is consistent, however, with the attitude of Peano's collaborators towards building models that the interpretation be made with the "ideas" we already have. In 1906, Peano defined a model for his arithmetic axioms using some work which had had a different aim (we shall mention this later), but remarking at the same time that "proof that the system of axioms for arithmetic, or for geometry, do not involve contradictions is not, to my mind, necessary. For we do not create axioms arbitrarily, we rather assume the simplest propositions we find to be as axioms, whether explicitly written or implicit, in every treatise of arithmetic or geometry" [Peano 1906, 365].

What Wilson could not be aware of is that there were also reservations about Padoa's attitude inside Peano's school but these were only to be expressed in the immediately ensuing years. Mario Pieri (1860-1913) indirectly disagreed with Padoa by affirming at the end of his 1904 paper that "it is vain to seek a direct and absolute proof of the compatibility of the arithmetical axioms in the field of Arithmetic itself" [Pieri 1904, 331, my translation]. Pieri

on the contrary looked at Hilbert's effort to give a logical proof of *compatibility* with interest although he remained conscious of the difficulties of avoiding the notion of number.

In 1906 Pieri came to believe in the possibility of a logical proof of compatibility with a logic that includes the concept of class:

je me propose précisément d'établir la *compatibilité des axiomes arithmétiques* de R. Dedekind et G. Peano *dans un domaine Δ de Logique pure* [...] en raisonnant dans les limites de la Logique des classes. [Pieri 1906, 203]

Pieri started from Burali-Forti's proof that finite classes are a model of Peano's axioms but he had to modify this by substituting II below to a Burali-Forti axiom that A. N. Whitehead (1861-1947) had pointed out as erroneous, and [Poincaré 1908, 209] had unfairly laughed at.⁷ Pieri's proof succeeds by relying on the logical axioms of the *Formulario* and the two "principes suivants:

- I. *Il y a au moins une classe infinie* (Le Tout est une classe infinie);
- II. *Étant donnée une classe infinie, dont les éléments sont à leur tour des classes, la classe formée par tous les éléments de celles-ci est elle-même infinie.*" [Pieri 1906, 207]

Pieri maintains that he can include these two principles without scruple among the logical axioms "car je n'y vois qu'une *détermination* convenable des concepts de *classe* et *représentation*" [Pieri 1906, 207].

But, independently of a discussion of this conviction,

je crois avoir établi que le concept de nombre entier, avec ses propriétés fondamentales (y compris le *principe d'induction*) peut être construit sur la Logique des classes de M. Peano, au moyen des propositions I et II. [Pieri 1906, 207]

Wilson would have appreciated Pieri's attempt which was the only piece of true logicism which came from Peano's school. For now, he continues to speak highly of the Italians' contributions as known to him:

Anyone who is acquainted with the articles presented to the Philosophical Congress at Paris in 1900 by Peano, Burali-Forti, Padoa and Pieri,⁸ cannot be convinced that these authors had become deadened, and the artificiality of their system is by no means so certain as it might be. (78)

Again at the end of the review in the bibliographic suggestions for further reading, Wilson suggests that these papers should be read.

7. For more details on this comedy of errors, see e.g., [Lolli 2012, x and xx-xxi]. [Burali-Forti 1896] is relevant also for the history of the axiom of choice, see [Moore 1982, 129].

8. [See references, for all the four of them identifiable by the year 1900.]

Since then, our author, Russell, has simplified and improved the older work of C. S. Peirce on the theory of relations, adapting it to the system of Peano, and has produced a coherent treatment of the great problems underlying mathematics. (78)

To him those papers “show the point at which the Italian school had arrived in 1900. It is since that time that most of Russell’s technical work has appeared” (93).

Wilson fails to perceive that the handing of the baton from Peano to Russell did not really constitute the launching of a project to take matters further but that there was a hiatus, a different conception. Peano was no logicist. He wanted to express existing mathematics in a rigorous, compressed and complete way. He obviously had to deal with arithmetic, geometry, algebra, real and complex numbers, calculus, but did not want to define mathematical entities (“numbers cannot be defined” he said in [Peano 1891]) and once the *Formulario* was on the right track he had only to make it grow, up to [Peano 1908]. As for foundations, “[h]ic difficultas maxime ex sermonis ambiguitate oritur”, difficulties come from the language ambiguities. The solution is the same as that suggested by Leibniz, namely to assign signs to the simplest ideas from which all others are composed by logical operators.

That date in 1900 also marks the point at which the Italian school practically ceased to be effectively involved. Wilson could probably not have had the historical perspective to see the school had run its course but contemporary mathematicians felt that no further contribution could come from it. Apart from the work of Pieri mentioned above which concerns questions of set theory which were in any case alien to Peano’s vision, the only breath of life in the following years is limited to [Peano 1906] and his defence of non predicative definitions against Poincaré.

In his 1906 paper on the Cantor-Bernstein theorem Peano answered Poincaré’s question as to whether the theorem could be proved without recourse to the natural numbers. After a general discussion of the impossibility of eliminating all mathematical ideas from such proof, otherwise only the logical signs would remain—showing clearly on this occasion that he is no logicist—Peano remarks that in this case one may avoid numbers.

The incriminated definition occurring in the known proofs was of the form

$$Z(u) = \bigcup \{g^n(u) : n \in \mathbb{N}\},$$

g being a bijection, which Peano substituted by

$$Z(u) = \bigcap \{v : u \subseteq v \text{ . } g^{\circ}v \subseteq v\}.$$

Then “suppose that only logic, and not arithmetic, is known, so that the symbols 0 , N_0 and $+$ are meaningless”; if u is not empty, and one of its elements is indicated with 0 , if we let $N_0 = Z(\{0\})$ and $x+ = g(x)$,

[and] I read 0, N_0 , + as in arithmetic [...] we deduce theorems identical with the axioms of arithmetic. [Peano 1906, 364–365, my translation],

just to add immediately however that there is no necessity for a model.

In any case, in 1908 Wilson still did not have the sensation that Peano's work was finished and that Russell had taken on a new way of working and could thus repeat that “reading modern Italian is a necessary condition”, to become symbolic logicians [Wilson 1908a, 188]. And perhaps this was not enough and Latin was also required as Wilson warns elsewhere “if I understand his inflexionless latin” [Wilson 1908b, 437].

3 Russell's logicism

Now Wilson turns to Russell, although he will have the opportunity to come back to Peano with pertinent remarks.

In section 3 of the review, *Reason*, Wilson briefly discusses how mathematics had needed to push its foundations back until they rested solely on logic. Mathematics as well as any reasoning obviously presupposes a mind capable of ratiocinative processes. It is usually assumed that if we are careful enough there is no need for formulating and learning the laws of thought before beginning to reason or even that a formulation of those laws is impossible.⁹ However the review began by talking of errors. Where do errors creep in? It is interesting to read the ideas of someone who was still near the beginning of the movement of arithmetization and introduction of epsilon proofs almost to the point of having experienced its bewildering novelties in his lifetime and who saw it as something still meaningful for foundations.

Where then do the errors creep in? An examination of some typical cases shows that it is generally through lack of a sufficiently careful definition of the terms. [...] In mathematics it is the absence of precise definition which brings in the erroneous statements concerning differentiation, continuity and infinity, with a host of others. The perception of this difficulty was the origin of the principle of arithmetization and of epsilon proofs. (79)

Finally, after one has mastered the principles of modern analysis, there is no need for the actual presence of epsilon in the proofs.

Nevertheless it is a satisfaction to have this formal method to fall back on whenever challenged by one's own hesitancy or by that of others. In like manner, who has not at times during some

9. He refers to “[J.-M.C.] Duhamel, *Des méthodes dans les sciences [de Raisonnement]*, [Gauthier-Villars], Paris, 1875, vol. 1, p. 17”.

long complicated or indirect logical demonstration felt the least bit uncertain; who would not be glad to have at his hand some formal method such as Peano's, based upon certain rudimentary propositions and concepts?

In truth, it is a matter of more consequence than is sometimes thought, to have clearly in mind those processes which are definitely to be admitted as logical. [...]

The question then becomes of fundamental importance: What is at the bottom of our logic? (79)

"We constantly use *propositions*, passing from certain propositions as hypotheses to certain others as conclusion" (80); looking for other principles "we come upon *classes* or sets of objects represented in ordinary speech by common nouns"; finally we perceive that the "*relations* are of the utmost importance" ("one of the lasting services of Russell" (80, fn.*).

The complete logical calculus, as now used, is a combination of these three types. (80)

We may grant, then, that logic is *necessary* to mathematics. It is affirmed to be *sufficient*. This in reality is the remarkable content of the definition [of pure mathematics] given by the author. (80)

But the affirmation of its sufficiency fully justifies and even renders imperative a critical examination of its principles that the simple necessity might never force us to. The number of logical premises which are sufficient to establish the calculus in all generality necessary for mathematics is small. However there are a certain number of elementary ideas such as implication, the notions of proposition, class, and relation that must be known.

It is the discussion of these questions which are of a philosophical rather than mathematical nature that fills the first Part of Russell's Principles. (80)

Notwithstanding the novelty, and the many philosophical and mathematical difficulties, Wilson has no doubt that "to a large extent the author is successful in his attempt" (80).

4 Some methodological notions

The next section, *Some notions*, is dedicated to elucidating certain technical notions to avoid the inconvenience of having to include definitions throughout the rest of the speech.

"Axiom" is a word that is best abandoned in pure mathematics. Its more familiar meaning, as "self-evident truth", has no place in pure mathematics, which is "a formal subject over which formal and not material implication

reigns.†”¹⁰ The proper word to use for statements we posit in mathematics would seem to be “postulate”.

What self-evident truths can there be concerning objects which are not dependent on any definite interpretation but are merely marks to be operated upon in accordance with the rules of formal logic? Postulates, however, may be laid down at will so long as they are not contradictory. It is the postulates which give the objects their intellectual though not physical existence. (81)

“Definition” for philosophers stands for “a process of analysis and exemplification which brings before the mind a real consciousness of the object defined” (81–82). Mathematical definition is the attribution of a name to some object whose existence has been established or postulated. “It is the process of replacing a set of statements by a single name and is resorted to solely for convenience” (82). Footnote* (82), quotes [Peano 1900] to the effect that it might be better to exclude a large number of definitions.

All definitions are nominal. But there are three styles of definition that can be illustrated in connection with the theory of integers.¹¹ One possibility for defining the class of integers is to find a class, whose elements are classes or propositions, among which there is an element analogous to zero (e.g., the null class), and it is possible to define operations with the properties we use for integers. We could say that this class is the class of integers. It would be a particular but satisfactory definition of integers. We would also be sure that there would be no contradictions in our system of integers unless there were a contradiction in our logic. Another possibility would be to write a suitable set of postulates with the appropriate symbols. “In order to prove the noncontradictoriness of our system of postulates and indefinables, that is, the existence of our elements” (83),¹² we should build some system which provided an interpretation of the indefinables and of the postulates, and in the end we should resort to our logic and thus there would not be much difference from the previous case. The definition would be more general in that the integers would not be a particular set but any set satisfying the postulates. Thirdly we could define, as Russell does, numbers by means of the property of having the same number in case a one-to-one relation exists between the elements. Also this kind of definition becomes a nominal one thanks to Russell’s work on relations.

Finally, since the use of postulates is so common, although in principle avoidable in pure mathematics, Wilson expresses “a few words” on consistency, independence, irreducibility and completeness. After mentioning Padoa’s

10. In † it is admitted that in logic “we should incline to use the word axiom [...] for we are dealing with the actual (mental) world and not with a system of marks. The basis of rationality must go deeper than a mere set of marks and postulates”.

11. Although implicit, the reference here is to [Burali-Forti 1901].

12. Notice how Wilson independently states Hilbert’s famous thesis on the connection between consistency and existence.

method for proving the redundancy of a symbol in terms of others, he observes that “Huntington[†] seems to have been the first to bring to effective use the idea of completeness” (84); with this “we have arrived at the limit of present ideas concerning the interrelations of the notions at the base of mathematics as defined by postulates” (84). These ideas were well known and discussed in the US, paving the way in fact for the clarification of the concepts of completeness and categoricity. “Completeness” was the translation of Hilbert’s *Vollständigkeit* and it was used in the sense of completeness in its semantical form, meaning that every sentence of the language of an axiomatic theory was either a logical consequence of the axioms or was incompatible with them, or as deductive completeness. It was also used for what came to be called categoricity, together with other words.

Edward Vermilye Huntington (1874-1952) in [Huntington 1902] had provided a set of postulates for the continuous magnitudes that he believed to be “complete”, meaning that postulates were mutually independent, non-contradictory and “sufficient”. By this he meant that “there exists essentially only one” model. “Only one” is intended through the modifier “essentially” to mean the same thing as our “up to isomorphisms”, although the word was not available.

In 1904 Oscar Veblen (1880-1960) followed advice from John Dewey (1859-1952) to suggest the term “categoricity” to substitute the ambiguous “completeness” [*Vollständigkeit*]. Veblen maintained he had the right to apply the undefined terms point and order to any class of objects satisfying the axioms but that he also aimed to show that “there exists *essentially one* such class” [Veblen 1904, 346]. Completeness, which he stated in semantical terms, would follow.

In 1905 Huntington borrowed Veblen’s term “categorical” for “sufficient”, conceiving it in the sense that any proposition in primitive language is either deducible from the postulates or contradictory to them (deductive completeness). “We have to admit however that our command of the processes of logical deduction is not yet, and probably never will be sufficiently complete to justify this assertion” [Huntington 1905, 210†].

Wilson must have known Veblen’s proposal since he uses the term “categoricity”. He had the opportunity to come back to categoricity in 1908 by intervening in the debate on Zermelo’s axiom of choice which he saw rather as a logical axiom. He is thus led to discuss logic in general. First he observes that

It is not always desirable and indeed not always possible to obtain a set of postulates which shall be categorical: for it may well happen that the systems to be determined are such that not even a one to one correspondence between their elements is available, to say nothing of the preservation of the interpretation of the symbols [*S*]. [Wilson 1908b, 434]

Then he recalls that

Huntington gives a subsidiary definition or explanation of the idea of categoricity wherein he asserts that if a set $[P]$ of postulates on the undefined symbols $[S]$ is categorical, then every proposition concerning $[S]$ must be deducible from the postulates $[P]$ or be in contradiction with them. [Wilson 1908b, 434]

In a system determined categorically, every proposition phrased in terms of $[S]$ is either compatible or incompatible with $[P]$.

What, however, does the word deducible mean? The meaning is entirely relative to the system of logic which is available for drawing conclusions from the set of primitive propositions $[P]$. Some may consider that the human mind has instinctively at its disposal all valid methods of deduction. This is a tremendous postulate, and one entirely devoid of other than sentimental value. In fact, if it leads to the abandoning of the research for valid methods of deduction, it is dangerous and worse than useless. It is an essential of the modern attitude in logic that the deducer should state distinctly his form of inference. Hence deducible cannot be regarded as equivalent to compatible.

It is clear that in an ideal perfection of logic compatibility and deducibility would be equivalent for categorically defined systems. That state of perfection appears at present to be very remote. [Wilson 1908b, 436]

5 *The Principles*

The next two sections, 5 *Numbers* (85) and 6 *Geometry and Mechanics* (87), are entirely devoted to the mathematical development of the *Principles*. The definitions of numbers, of cardinals and of ordinals are obviously discussed, “with the guidance of the principle of abstraction” (85) (meaning: equivalence classes). Finite and infinite are neatly distinguished. According to Wilson, an advantage of Russell’s method, with contributions by Alfred North Whitehead (1861-1947), “is that by the use of logical addition the numerical addition of a finite or infinite number of finite or infinite cardinals may be and indeed (if we invoke the principle of abstraction) should be defined in such a manner that the order in which the numbers are added plays no part” (85). To Wilson,

This is a great victory for common sense and must appeal to everyone as a vindication of the school-child in his inherent notion that he has the same number of marbles whether he has five in one pocket and three in another or three in two pockets and two in a third, no matter which of his pockets these be. The principle of commutation and association of the terms in addition is entirely done away with, except in so far as mechanical difficulties prevent

us from writing simultaneously a number of terms and the signs of addition connecting them. (85)

Of course, “there still remain difficulties to solve”, but “there is no reason why he [the author] should not find adherents who will take up the work and attempt the solution in a spirit of hearty cooperation” (87).

As for real numbers,

[t]here is a school of creationists who, when they find that certain infinite processes lead to no rational limit nor yet to a number which becomes infinite, postulate the existence of a limit and thus obtain the irrational numbers. The author does not consider an *ipse dixit* like this to be a sufficiently good theorem of existence. He therefore considers infinite sets of rationals and by means of them he forms a set of things which he calls real numbers. A real number is neither a rational nor an irrational; it is a certain infinite set of rationals. (87)

Wilson appreciates the “very satisfactory account of the philosophy of the infinite and of the continuous”.

The comments on geometry are interesting in that Wilson sees Russell’s presentation as a good example of how to avoid the postulates method. He skips however the details of the definition of geometry as “the study of series [successions] of two or more dimensions” (87) so that readers of the review cannot grasp it unless they resort to Russell’s book. The simplest example is a succession of reals. The anomalous definition is partly due to the fact that “Mathematical geometry has long since been divested of all spatial relations between its elements” (87). On the other hand, as those who define geometry by postulates are forced to show the existence of their elements by having recourse to systems of numbers the question is quite pertinent:

Why not begin with a purely nominal definition like the above and avoid the trouble of proofs of existence, of independence, and of irreducibility? (88)

Wilson confronts the new approach with the previous one of [Russell 1897] whereby geometry depended on mechanics, and shows the path leading to its foundations pursued by the author as deeply as to the logical basis before returning to rebuilding it through the solution of many logical issues.

6 Conclusions

One conclusive remark on logic, in section 7 *Conclusions* (90), is the warning that there are many systems of logic at present. Since a few of the topics which need a logical treatment are very complex like infinity and the continua, “it

might not be regarded as surprising if some points were found to stand out permanently, so that logicians will permanently disagree" (90). In fact Wilson is aware that there is a logical difficulty in the very logical system developed by Russell. Frege's repair work is discussed in the appendix to the book under review. In (90, fn.*), Wilson recalls Hilbert's attempt in the Heidelberg address (quoted before) to recast the principles of logic and arithmetic so as "to render them sufficient for mathematical reasoning":

We certainly hope he has succeeded in doing so to the satisfaction of both mathematicians and philosophers. (90, fn.*)"

Meanwhile

[...] [i]t is dangerous to accept the naïve point of view of those who claim that a certain piece of reasoning depends on the operation of logic alone but who fail to state what those operations are. (90)

From the pedagogical standpoint, Wilson entirely accepts Poincaré's warning that "pure logic alone [...] is harmful to the earlier development of the mind" (91). Hence, instead of troubling students with elaborate deductions of the properties of arithmetic operations, it would be better to let them appreciate the ideas of finite and infinite cardinals and ordinals, of compactness and continuity, the different kinds of infinity. A clear-cut "physical conception" that numbers possess order and may be associated with the points of a line is both necessary and sufficient for ordinary rigorous analysis.

An inadequate vague idea regarded as a useful working hypothesis seems, on the whole, productive of more good and less harm than an inadequate definite idea regarded as final. (91)

As for mathematics, "we have learned that many of the objects which have been thought of as individual must be regarded as classes" (91). According to Wilson, "we cannot define Euclidean space, but we can define the class of all Euclidean spaces". In (91, fn.[†]), he notes that this apparently lowers the importance of the concept of completeness (discussed before in relation to Huntington):

For it appears as if the one-to-one correspondence between the different Euclidean spaces were really of minor significance. This is but another instance of the fact that the elements themselves are unimportant—that it is the abstraction from them which is most fundamental. (91, fn.[†])

The idea of completeness is however a step forward toward a fuller description of the systems dealt with. Wilson would certainly be happy with the many later concepts of model theory.

A note of optimism in conclusion. Since during the construction of the mathematical objects "we have introduced no new indefinables, no new

postulates, no processes other than those of logic, there is no possibility of our arriving at contradictions except through the failure of our logical system to be logical; and behind this we cannot go" (91). The existence of the classes which we have dealt with remained to be shown and this was done by Russell, starting from below—the null class, the finite cardinals, the class of finite cardinals, and so on.

The promised second volume will contain "actual chains of deduction leading from the premises of logic through arithmetic to geometry" (92). Wilson could hardly surmise that this logic would have to wait till 1908.

Finally: "For those who wish sooner to get at the Peano-Russell point of view", a short bibliography is added which is "very incomplete" but useful to trace the development of the idea. Beyond the papers quoted in the footnotes, it comprises Peano's two 1889 seminal works, "the starting point of the whole movement", the four Italian Paris contributions again, a paper by Whitehead on cardinal numbers and one by Burali-Forti on a general theory of numbers. *Logica matematica* by Burali-Forti in the series of Manuali Hoepli may serve as a textbook.

The *Formulaire de mathématiques* edited by Peano, is rather hard to begin on. The *Rivista de matematica* [...] also edited by Peano, furnishes much easy and instructive reading matter. (92)

As for our conclusions, it seems to us that when talking of the "Peano-Russell point of view", Wilson's assessment is even too generous, but after all fair, when he ascribes to Peano "an intimate formal relation between logic and all mathematics [...] precursor to a yet more intimate philosophical relation" (76), left to be revealed by Russell. It is not the case that "he [Peano] was always more precise than anyone else, and that he invariably got the better of any arguments upon which he embarked [at Paris 1900 Congress]" as Russell said. In Russell's exalted words, Peano's logic was "a new mathematical technique", as stated in [Russell 1967, 144–145]. Peano's logic used classes, and, although he himself probably lost interest in its development, certain of his followers at least [Pieri 1906] paid attention to the necessity of axioms for classes and [Burali-Forti 1916a,b] later gave a definition of ordered n -tuples.

Bibliography

BURALI-FORTI, Cesare [1896], Le classi finite, in: *Atti della Reale Accademia delle Scienze di Torino*, vol. 32, 34–52.

— [1901], Sur les différentes méthodes logiques pour la définition du nombre réel, *Bibliothèque du Congrès Inter. de Philosophie*, 3, 289–307, doi: 10.5840/wcp11901311.

- [1916a], Sulla definizione di coppie, terne, ecc., in: *Atti dell'Accademia dei Lincei: Rendiconti*, vol. 25, 405–413.
- [1916b], Ancora sulla definizione di coppie, terne, ecc., in: *Atti dell'Accademia dei Lincei: Rendiconti*, vol. 25, 206–207.
- COUTURAT, Louis [1901], *La Logique de Leibniz*, Paris: Alcan.
- [1904], Review of [Russell 1903], *Bulletin des sciences mathématiques*, 28, 129–147.
- GRATTAN-GUINNESS, Ivor [2000], *The Search for Mathematical Roots, 1870–1940*, Princeton: Princeton University Press.
- HILBERT, David [1900a], Mathematische Probleme, *Nachrichten von der Königlischen Gesellschaft der Wissenschaften zu Göttingen*, 3, 253–297.
- [1900b], Über den Zahlbegriff, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 8, 180–184.
- HUNTINGTON, Edward V. [1902], A complete set of postulates for the theory of absolute continuous magnitude, *Transactions of the American Mathematical Society*, 3(2), 264–279, doi: 10.2307/1986422.
- [1905], A set of postulates for ordinary complex algebra, *Transactions of the American Mathematical Society*, 6(2), 209–229, doi: 10.2307/1986300.
- IRVINE, Andrew (ed.) [1999], *Bertrand Russell: History of Philosophy, Ethics, Education, Religion and Politics*, vol. 1, London: Routledge.
- LOLLI, Gabriele [2012], Cesare Burali-Forti (1861-1931) e la logica matematica, in: *Logica matematica*, edited by C. Burali-Forti, Pisa: Edizioni della Normale, vii–lvii.
- MOORE, Gregory H. [1982], *Zermelo's Axiom of Choice*, New York: Springer.
- PADOA, Alessandro [1903], Le problème n.2 de M. David Hilbert, *L'Enseignement mathématique*, 5(1), 85–91, doi: 10.5169/seals-6625.
- PEANO, Giuseppe [1889], *I Principii di geometria logicamente esposti*, Turin: Fratelli Bocca, reprinted in [Peano 1958, 56–91].
- [1891], Sul concetto di numero. Nota I, *Rivista di Matematica*, 1, 87–102, reprinted in [Peano 1959, 80–109].
- [1900], Les définitions mathématiques, in: *Compte rendu du deuxième Congrès int. des mathématiciens*, Paris: Gauthier-Villars, 313–331, reprinted in [Peano 1959, 362–368].

- [1906], Super theorema de Cantor-Bernstein, *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, 21(1), 360–366, doi: 10.1007/BF03013483, also in *Rivista di matematica*, vol. viii, n. 5, 1902-1906, 136–143; reprinted in [Peano 1957, 337–344].
- [1908], *Opere Scelte*, vol. I, Rome: Cremonese.
- [1957], *Formulario mathematico*, vol. V, Turin: Fratelli Bocca, reprinted in 1960, Rome: Cremonese.
- [1958], *Opere Scelte*, vol. II, Rome: Cremonese.
- [1959], *Opere Scelte*, vol. III, Rome: Cremonese.
- PIERI, Mario [1898], I principii della geometria di posizione composti in sistema logico-deduttivo, *Memorie della Reale Accademia delle Scienze di Torino (serie 2)*, 48, 1–62, reprinted in [Pieri 1980, 101–162].
- [1899], Della geometria elementare come sistema ipotetico-deduttivo. Monografia del punto e moto, *Memorie della Reale Accademia delle Scienze di Torino (serie 2)*, 49, 173–222.
- [1904], Circa il teorema fondamentale di Staudt e i principii della geometria proiettiva, in: *Atti della Reale Accademia delle Scienze di Torino*, vol. 39, 313–331, reprinted in [Pieri 1980, 289–308].
- [1906], Sur la compatibilité des axiomes de l'arithmétique, *Revue de Métaphysique et de Morale*, 14(2), 196–207, doi: 10.2307/40893254.
- [1980], *Opere sui fondamenti della matematica*, Rome: Cremonese.
- POINCARÉ, Henri [1903], Review of Hilbert's Foundations of Geometry, *Bulletin AMS*, 10(1), 1–23, doi: 10.1090/S0002-9904-1903-01061-1, Engl. transl. of “Les fondements de la géométrie”, 26(2), 249–272.
- [1908], *Science et Méthode*, Pris: Flammarion.
- RUSSELL, Bertrand [1897], *An Essay on the Foundations of Geometry*, Cambridge: Cambridge University Press.
- [1901a], *Essai sur les fondements de la géométrie*, Paris: Gauthier-Villars, French transl. by A. Cadenat.
- [1901b], Sur la logique des relations avec des applications à la théorie des séries, *Rivista di Matematica*, VII, 115–136; 137–148, also in English in [Russell 1956, 1–38].
- [1903], *The Principles of Mathematics*, Cambridge: Cambridge University Press.

- [1956], *Logic and Knowledge*, London: Allen & Unwin, edited by R. C. Marsh.
- [1967], *The Autobiography of Bertrand Russell 1872-1914*, London: Allen & Unwin.
- VAILATI, Giovanni [1891], Le proprietà fondamentali delle operazioni della Logica Deduttiva studiate dal punto di vista d'una teoria generale delle operazioni, *Rivista di Matematica*, 1, 127–134.
- [1892], Sui principî fondamentali della Geometria della Retta, *Rivista di Matematica*, 2, 71–75.
- VAN HEIJENOORT, Jean (ed.) [1967], *From Frege to Gödel, a Source Book in Mathematical Logic, 1879–1931*, Cambridge: Harvard University Press.
- VEBLEN, Oscar [1904], A system of axioms for geometry, *Transactions of the American Mathematical Society*, 5(3), 343–384, doi: 10.1090/S0002-9947-1904-1500678-X.
- WILSON, Edwin B. [1904], *The Foundations of Mathematics*, *Bulletin of AMS*, 11(2), 74–93, doi: 10.1090/S0002-9904-1904-01185-4, reprinted in [Irvine 1999, 4–20].
- [1908a], Review: *Symbolic Logic*, *Bulletin of AMS*, 14(4), 175–191, doi: 10.1090/S0002-9904-1908-01573-8.
- [1908b], Logic and the continuum, *Bulletin of AMS*, 14(9), 175–191, doi: 10.1090/S0002-9904-1908-01652-5.