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Conceptions of Continuity: William Kingdon Clifford's Empirical Conception of *Continuity* in Mathematics (1868–1879)

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Résumé : Le concept de continuité est fondamental pour l'analyse mathématique contemporaine. Cependant, la définition actuellement employée, apparemment bien fondée de ce concept, n'est que l'une des nombreuses versions historiquement énoncées, utilisées et affinées par les mathématiciens au travers des siècles. Cet article présente la façon dont William Kingdon Clifford (1845-1879) a façonné ce concept en lui donnant des bases physiques. La présentation de l'effort de Richard Dedekind (1831-1916) pour établir mathématiquement cette notion dans une perspective conventionnaliste permettra de mieux apprécier la spécificité de chacune des deux démarches. Cette étude montrera quel rôle historique la continuité a joué dans plusieurs projets mathématiques et philosophiques, et comment elle a été façonnée à partir des différents centres d'intérêt de ses utilisateurs.

Abstract: The concept of continuity is fundamental to contemporary mathematical analysis. However, the seemingly well-founded definition currently used for this concept is just one of many versions that have been historically proffered, used and reshaped by mathematical practitioners over the centuries. In looking at the particular manner in which William Kingdon Clifford (1845-1879) defined and negotiated the physical underpinnings to "continuity" in his works, and by comparing that account to Richard Dedekind's (1831-1916) conventionalist definition of the concept, it becomes clear that "continuity" has served differing interests over the centuries and has been shaped by the process of applying it to various philosophical and mathematical projects.

Introduction

In the edited collection *Conceptions of Ether: Studies in the history of ether theories* [1981], G. N. Cantor and M. J. S Hodge write that, in presenting a volume composed of various case studies in the history of ether concepts, their

book introduces readers “to the broadest themes in the scientific thought of the eighteenth and nineteenth centuries” [Cantor & Hodge 1981, ix].¹ Likewise, histories of “continuity” can introduce historians to the broadest themes in eighteenth- and nineteenth-century mathematics. “Continuity” is one of the fundamental concepts in calculus, and its history is by no means simple or straightforward. The aim of this paper is to show how one mathematician, namely William Kingdon Clifford (1845-1879), conceived of mathematical “continuity”, how he used it, and how he subtly redefined it as part of his grander philosophical project—to prove that scientific theories based on action-at-a-distance principles (*i.e.*, instantaneous action across excessively large or infinitesimally small expanses of space) constitute poor means of explaining physical phenomena.

As a mid-19th century empiricist—influenced as much by the ideas of Charles Darwin and Herbert Spencer as he was by the ideas of Bernhard Riemann, Sir William Rowan Hamilton and Hermann Grassmann—Clifford’s approach to continuity can be summarized as follows: continuity is a conventional mathematical tool based on empirical evidence. Its mathematical definition is an abstraction of the assumed existence of continuity in space and time. *Continuous space* implies that there are no gaps or moments of non-existence in the fabric of the universe; *continuous time* implies that there are no gaps in the fabric of forward-moving time. Specifically with regards to the physical aspect of continuity, Clifford invoked the image of a continuous medium that pervades the entire universe (*i.e.*, an ether).

Thus, for Clifford, the techniques of calculus are conventional tools that describe phenomena in space. The accuracy of the descriptions gained from the use of such tools (for example, descriptions depicting the rate of acceleration of an object falling towards Earth) is fundamentally dependent upon the

1. In their account of ether histories, Cantor and Hodge identify five categories of “ether” concepts that have been discussed and used by natural philosophers, mathematicians, and physicists over the course of the past century. The authors write: “We might introduce the historical study of ether theories by stating simply what ether is. But two reflections should make us pause before attempting this. First, although essentialism is now favoured again in reputable quarters, the lessons of Wittgenstein and others still stand. For many kinds of things it is futile to seek a common and distinctive essence. For many terms it is misleading to demand a definition specifying the conditions necessary and sufficient for their application. So it is with *ether*. We cannot usefully indicate properties that all ethers must and only ethers can have. Second, even if we could, we should not wish to make the attempt independently of historical inquiry. For definition demarcation of one kind of theory from others may serve merely to separate lines of theorising that in fact developed together, or to conflate those that developed apart” [Cantor & Hodge 1981, 1]. I could repeat Cantor’s and Hodge’s reflections here as an introduction to my own account of “continuity” as it appeared throughout the 19th century and earlier. “Continuity” was not a clearly defined mathematical notion. It was not a perfectly understood physical notion. It was used by various authors in various ways serving their varying interests. In short, it is a concept with a history, and the historical construction of its meaning is still ongoing.

truth of the assumption that continuity exists in the ontological structure of the universe. Following in the footsteps of a quintessentially British-empiricist tradition, Clifford accepted that there is no way to know certainly whether the universe really is continuous; he acknowledged that many physical phenomena might in fact be the product of *discontinuities* in space, rendering analytical techniques useless and inappropriate. However, Clifford contended that if scientists correctly adopted the assumption that continuity is true of the structure of the universe (as Clifford himself believed it to be), then they must avoid the notion of “force” as a causal explanation of phenomena. Forces, by their very nature, are *α*-physical; they exist independently of the material bodies they act upon. Newtonian gravity is a case in point—the belief that two bodies simultaneously express an attraction towards one another implies that the “force” of attraction operates across immense tracks of space instantaneously and thereby causes bodily motion. Such action amounts to a “discontinuity” in space, meaning that physical theories often devolve into metaphysical speculation. Clifford argued that scientists and mathematicians who accurately accept the implications of analytical techniques must accept certain limitations to their scientific theorizing. They must accept that *material interactions* alone can result in changes of motion. For example, continuity implies that “energy” (then a newly emergent concept in physics) can only pass from one piece of matter to another; it cannot be transferred through a void.

Clifford’s contentions with regards to continuity would not have been out of place among many of his contemporaries in the 1870s, given that many of his colleagues sympathized with the empiricist undertones in his foundational arguments. However, a mathematical philosophy of the type that he was advocating was not easily accepted by the whole of the mathematical community either. Indeed, other mathematicians were interpreting “continuity” in radically different ways at the same time that Clifford was pushing his empiricist agenda. This is demonstrated in Richard Dedekind’s (1831–1916) account of continuity. Dedekind and Clifford make useful comparative case studies because both mathematicians clearly defined their guiding conceptions of “continuity” by the mid-1860s and both mathematicians published their views on continuity primarily in the 1870s. Both practitioners were also deeply interested in exploring the degree to which mathematical “continuity” was necessarily reliant upon physical structures for its ontological content. In addition, both Dedekind and Clifford were influenced by Bernhard Riemann (1826–1866), who was a colleague of Dedekind’s, and whose work in non-Euclidean geometry was of fundamental importance to Clifford’s mathematical research. As a result, the mathematicians wholeheartedly adopted many of Riemann’s conventionalist musings on the foundations of mathematics.

Yet, for all of their similarities, Dedekind and Clifford also *diverged* significantly in terms of their eventual definitions of “continuity”. For Dedekind, mathematical “continuity” is made precise by appealing to infinitesimally small intervals on a number line. Thus, “continuity” requires a formal definition in mathematics that is independent of physical truths in the universe. Dedekind

defined “continuity” through the use of the mathematical concept known as an “infinitesimal”. He argued the “infinitesimal” is not based upon spatial or geometrical intuition. On Dedekind’s view, the infinitesimal measuring stick that mathematicians embody in the notion of a “limit” can be rendered meaningful through purely mathematical (*i.e.*, arithmetical) construction. Conversely, Clifford argued the “infinitesimal” is fundamentally and irrevocably empirical; it cannot be understood without appealing to descriptive properties of material space.

In this paper, I will explore the formal mathematical definition of continuity that Dedekind provided in his account of real numbers (initially written in the 1850s, but not translated into English until the 1870s) and compare it to Clifford’s own materialist account (published throughout the 1870s). The aim is to highlight the sociological claim that mathematical and scientific concepts, such as “continuity”, are best understood as bearing a family-like resemblance to a plethora of differing notions. “Continuity” has been defined in a plurality of ways by different practitioners throughout history, and it has been used to serve their differing interests.

1. On Continuity and Limits

From a contemporary standpoint, any first year course in calculus introduces students to the intuitive, yet formally elusive, concept of “continuity.” Consider for example the introductory remarks made in one such textbook. Here the student is told:

The study of two concepts essential for calculus [is] *limits* and *continuity*. Although the ideas behind them appear simple, they are in fact difficult to understand. (It is often the case in mathematics that the simplest-seeming ideas are the hardest.) It is easy to grasp the *idea* of a limit, and the *idea* of continuity. To say that an infinite sequence of numbers $x_1, x_2, x_3, \dots$, *approaches a number x as a limit* simply means that the further we go in the sequence, the closer we get to x , and we can get as close as we want simply by going far enough out in the sequence. Continuity is even easier—a function f is continuous on an interval $[a, b]$ if it is possible to sketch its graph over the interval without lifting the pencil from the paper. . . . Despite this intuitive simplicity, however, both ideas have proved extremely difficult to formalize in mathematical language. [Voorhees & Chun 2004, 79]

The student who follows such a lesson is usually then introduced to Zeno’s paradox in which Achilles and the tortoise race; the tortoise is given a head start and Achilles is never able to catch up because he must always travel half

the distance between himself and the tortoise *ad infinitum*. This is presented as the historical antecedent to the modern concept of “continuity”; the notion of a “limit” is introduced as the accepted resolution to Zeno’s paradox:

The mathematical resolution of the paradox is to note that the sequence of values $x_1, x_2, x_3, \dots$ converges to a limit x , and that this is the point at which Achilles passes the tortoise. Even today, however, philosophers (but not mathematicians) argue about whether this paradox has actually been resolved. [Voorhees & Chun 2004, 81]

A full account of the various contemporary accounts of continuity is beyond the remit of this paper. However, it is useful to note that within the world of calculus education, the now accepted “precise” account of continuity is as follows:

The limit of a function as x approaches a can often be found simply by calculating the value of the function at a . Functions with this property are called *continuous at a* . We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place gradually, without interruption or abrupt change.)

Definition: A function f is **continuous at a number a** if

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Notice that Definition 1 requires three things if f is continuous at a :

1. $f(a)$ is defined (that is, a is in the domain of f),
2. $\lim_{x \rightarrow a} f(x)$ exists,
3. $\lim_{x \rightarrow a} f(x) = f(a)$.

The definition says that f is **continuous at a** if $f(x)$ approaches $f(a)$ as x approaches a . Thus, a continuous function f has the property that a small change in x produces only a small change in $f(x)$. In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

If f is defined near a (in other words, f is defined on an open interval containing a , except perhaps at a), we say that f is **discontinuous at a** , or f has a discontinuity at a , if f is not continuous at a .

Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person’s height. But discontinuities do occur in such situations as electric currents.

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it. The graph can be drawn without removing your pen from the paper. [Stewart 2003, 102]

We see that the precise account of continuity (propositions 1, 2, and 3) relies, ultimately, upon physical or geometrical intuitions of uninterrupted space and time and the associated mental image of an infinitesimal interval. However, the language used to define it invokes formal symbols, suggesting an independence from physical concerns. Hence, the modern conception of “continuity” is, in many ways, a *mélange* of previously distinct approaches.

It is in rendering problematic the now accepted usage of “continuity” that historians and sociologists can begin to see heterogeneity in the historical definitions offered for this seemingly settled and homogenous concept. Indeed, mathematicians have used the word “continuity” to refer to very different things at different times. A quick scan of literature spanning the past 10 decades indicates that rarely has there existed a consensus over the meaning of “continuity” in mathematics, physics, or philosophy [Pitkin 1906]. Early 20th century discussions of “continuity” include, for example, Philip E.B. Jourdain’s 1908 account of “irrational numbers” in which he argued that “irrational numbers” are often taught to students by appealing to ambiguous and poorly defined notions of spatial extension [Jourdain 1908], while in 1933 G. J. Whitrow argued that the problem with relying upon the formal definition of limits in calculus is that the “continuum”, upon which it is based, is nothing but “an ingenious theoretical substitution for the vague empirical notion of the continuous” [Whitrow 1933, 157]. Jose A. Benardete later wrote that rational and real numbers are arithmetical *approximations* to the “continuum”—they are based on an intuitive, spatial sense of continuity as represented in geometrical magnitude [Benardete 1968]. Conversely, Ian Meuller argued in the mid-20th century that the “continuum” could, and should, be rendered intelligible in terms of discrete units *only*—not geometrical magnitudes. Meuller believed practitioners could develop new and independent notions of “continuity” without having to appeal to any geometrical or physical phenomenon as referents [Meuller 1969].

More recently, the question of “continuity” and its place in the history of mathematics has constituted the unifying theme for a colloquium held in France in 1990. That gathering resulted in the compendium entitled *Le Labyrinthe du Continu* [1992]. The editors recount:

La question du continu est à tout égard privilégiée : par son *ancienneté*, puisque, on le sait, la philosophie et les mathématiques venaient juste de naître en Grèce lorsqu’elle fut pour la première fois formulée ; par sa *permanence*, puisqu’elle n’a pour ainsi dire jamais cessé d’occuper le devant de la scène, suscitant des débats profonds, passionnés, incertains ; par sa *centralité* dans le champ

scientifique, puisqu'elle est aujourd'hui clairement un enjeu non seulement pour la logique et les mathématiques, mais aussi pour la physique et les sciences cognitives. [Salanski & Sinaceur 1992, III]

In other words, continuity's definition, use, and legitimate place within mathematical, physical and philosophical speculation are still open to considerable debate. The aim thus far has been to highlight the fact that debates over "continuity" have continued unabated despite a seemingly standardized view of the formal notion in mathematics.

2. Continuity and Dedekind's Approach

Dedekind was writing on "continuity" at a time when Clifford was finishing his undergraduate studies at Cambridge.² The two mathematicians were both deeply influenced by the works of Bernhard Riemann. Dedekind had been Riemann's contemporary as a student at Göttingen in the 1850s; Clifford would later become the translator and advocate of Riemann's work in Great Britain in the early 1870s. As a result of their common Riemannian background, both Dedekind and Clifford shared many fundamental beliefs regarding the nature of mathematical knowledge. Both, for example, thought that mathematical knowledge was constructed and conventional at root. Yet, they each defined "continuity" in notably different ways. It is worthwhile to briefly explore Dedekind's approach to the concept in order to observe the sociological claim that mathematicians draw on social resources (including philosophical belief systems) to interpret and reconstruct mathematical notions idiosyncratically, recreating the concepts they engage with along the way.

Largely due to his explicit essay on the topic, "Continuity and Irrational Numbers", which was conceived in the 1850s but not published until the 1870s, Dedekind usually comes to mind today when mathematicians think of the formalisation of continuity. Consider, for example, the opinion of Walter Pitkin, who states:

The problem of continuity enters into mathematics, physics and philosophy in one guise or another. It seems to have reached a satisfactory solution only in the first-named field, where the theory of continua of higher order, and the Dedekind theory of the nature of irrationals, appear to have brought about that much-desired freedom from paradoxes which the physicist and philosophical geometer cannot attain. [Pitkin 1906, 597]

2. In the Preface to his essay "Continuity and Irrational Number", Dedekind wrote that in 1858 he found himself "obliged to lecture upon the elements of the differential calculus," and he "felt more keenly than ever before the lack of a really scientific foundation for arithmetic" [Dedekind 1963].

Dedekind was among those who rejected the use of physical intuition and empirical observation in the behaviour of curves as the foundation for a mathematical conception of continuity.

As a former student of Gauss's at the University of Göttingen from 1850 to 1852, Dedekind completed his *Habilitation* at the same time as Bernhard Riemann in 1854. Following Gauss's death, he was mentored both by Riemann and J.P.G. Lejeune-Dirichlet, and later began to teach at Göttingen. By 1874, having moved to Zürich, Dedekind had become influenced by Cantor's attempts to introduce the "freedom of definition" into mathematics, after which he devoted most of his efforts to the development of set theory. The question of continuity and its links to infinitesimal analysis (as advanced by A.L. Cauchy in particular) was thus an early concern for Dedekind. Yet, from his introductory remarks to the essay indicated above, it is also clear that "continuity" remained a concern for him well into the 1870s.

In the preface to his work on irrational numbers, Dedekind wrote that his interest in the concept of continuity peaked when he was hired to teach calculus at the Polytechnic in Zürich in 1858. In appealing to "the notion of the approach of a variable magnitude to a fixed limiting value," Dedekind said he had recourse to "geometric evidences" which were highly useful from a didactic point of view. Yet, that "this form of introduction into the differential calculus can make no claim to being scientific, no one will deny," he argued [Dedekind 1963, 1]. Dedekind lamented that,

Even the most rigorous expositions of the differential calculus do not base their proofs upon continuity but, with more or less consciousness of the fact, they either appeal to geometric notions or those suggested by geometry, or depend upon theorems which are never established in a purely arithmetic manner. [Dedekind 1963, 2]

Dedekind's sole objective, therefore, was to find a "purely arithmetical and perfectly rigorous foundation for the principles of infinitesimal analysis" such that "continuity" would finally get a non-physical definition. Dedekind's belief was that a truly "scientific basis" for differential calculus and infinitesimal analysis is to be found not in the observation of geometric behaviour (of curves) or in any ambiguous spatial intuitions, but in arithmetic, where "arithmetic" is to be understood as the theory of discrete numbers.

For Dedekind, the "whole of arithmetic" is an extension of the simple act of counting and "counting itself [is] nothing else than the successive creation of the infinite series of positive integers in which each individual is defined by the one immediately preceding" [Dedekind 1963, 4]. When combined with the four basic arithmetic operations (addition, multiplication, subtraction, and division), the chain of these numbers proves to be an "exceedingly useful instrument." The series of positive *counting* numbers always allows for addition and multiplication; it does not, however, allow for subtraction and division in

all instances. It was this limitation that had propelled humans to “creatively” construct negative numbers and fractions in order to produce a more “complete” set of numbers (the set of rational numbers) which possesses two key characteristics:

- 1) Any of the four operations can be performed on any two elements in the system, except in the case of division by 0; and,
- 2) The system “forms a well-arranged domain of one dimension extending to infinity on two opposite sides” [Dedekind 1963, 5].

Dedekind admitted that the latter characteristic seemed to have a geometric underpinning. To avoid the “appearance” that “arithmetic was in need of ideas foreign to it,” Dedekind chose to focus on the “purely arithmetic” properties of the rational number system, which he claimed were dependent only upon the notion of a “cut” (*Schnitt*) [Dedekind 1963, 5]. The “cut” in question referred to the fact that for any rational number, a , the system of rational numbers divides into two sets: A_1 , and A_2 . “Every number of the first class A_1 is less than every number of the second class A_2 ,” he wrote [Dedekind 1963, 6]. An analogy could be made between a straight line with an arbitrarily chosen origin (o) and the rational number series. In such a model, every rational number is represented by one point on this line. On the straight line, L , “there [would be] infinitely many points which correspond to no rational number,” he noted [Dedekind 1963, 8]. Therefore, the straight line is infinitely “richer in point-individuals than the domain R of rational numbers in number-individuals” [Dedekind 1963, 9]. To account for all “phenomena in the straight line” (that is, to account for all “cuts” on the line) the mathematician must *construct* new numbers “such that the domain of numbers shall gain the same completeness, or as we may say at once, the continuity, as the straight line”³ [Dedekind 1963, 9].

Dedekind further noted that irrational numbers are often introduced with reference to the conception of “extensive magnitudes.” Such “magnitudes” have not, however, been “carefully defined” and pedagogical appeals to extensive magnitude mean that mathematicians have been explaining “number” as the product of a process of *measuring* [Dedekind 1963, 9]. Dedekind’s objective was to separate himself from that tradition in order to demonstrate that the basis for mathematical continuity is arithmetical; thus, it is not reliant upon measurement or upon the assumed existence of continuity in space, which underlies any process of physical measurement.

Historically, the comparison of the domain R (rational numbers) with a straight line has “led to the recognition of the existence of gaps, of a certain incompleteness or discontinuity of the former, which we ascribe to the straight line completeness, absence of gaps, or continuity,” Dedekind wrote [Dedekind 1963, 10]. Naturally, mathematicians have been left wondering what continuity actually refers to. “Everything must depend on the answer to this question”

3. The emphases are Dedekind’s own.

and “only through it shall we obtain a scientific basis for the investigation of *all* continuous domains” [Dedekind 1963, 10]. For Dedekind, the answer is not to be found in “vague remarks about the unbroken connection in the smallest parts” [Dedekind 1963, 10]. Rather, a precise account of continuity requires a definition of the following sort:

If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one [point] which produces this division of all points into two classes, this severing of the straight line into two portions. [Dedekind 1963, 11]

This tool can be understood to be a cut upon a line which is not represented by a rational number. Whenever we have a cut (A_1, A_2) , in which A_1 is the lower limit of the number and A_2 is the upper limit (neither of which is produced by a rational number) then mathematicians must create a new *irrational* number, which should be regarded as completely defined by the cut (A_1, A_2) . In effect, every irrational number becomes an interval on a line in which the upper and lower limits of the interval are the same.

To obtain a full definition of continuity, mathematicians also need to consider the relationship between two cuts (A_1, A_2) and (B_1, B_2) . If two cuts produce two numbers α and γ , then the principle of continuity indicates that there is an infinite number of different numbers β (that is, an infinite number of cuts or intervals) lying between those two numbers. The upshot of this definition is that in considering real number operations such as $\sqrt{2} \times \sqrt{3} = \sqrt{6}$,⁴ mathematicians must appeal to the idea of continuous functions and limiting values. For Dedekind, arithmetic “continuity” is one based on limiting values defined by infinitesimal intervals. In particular, if we have two numbers a and β , which are characterized by the properties that if ϵ is an arbitrarily small, positive magnitude then $x < a + \epsilon$ and $x > \beta - \epsilon$, but never $x < a - \epsilon$ or $x < \beta + \epsilon$. If a and β are different from one another, “However small be the positive magnitude ϵ , we always have finally $x < a + \epsilon$ and $x > \beta - \epsilon$ [where] x approaches the limiting value a ” [Dedekind 1963, 27]. These examples demonstrate the fundamental connection between the principle of continuity and infinitesimal analysis, Dedekind claimed; they also indicate the manner in which continuity can be arithmetically presented and thus severed from any geometrical roots.

Dedekind noted that although most of his readers would find this definition of continuity to be common sense, and although they might even be

4. In a celebratory account of Dedekind’s contribution to the field of number theory, David Fowler writes that mathematicians often have a “naïve belief that arithmetical operations on decimals pose no problems.” Fowler contends that Dedekind’s definition of continuity allows us to recognize that stating $0.999\dots = 1.000$ is problematical unless we first recognize and adopt a synthetic approach to the creation of new types of numbers. Such an approach is necessary, in Fowler’s view, for the resolution of the equation $\sqrt{2} \times \sqrt{3} = \sqrt{6}$, which presupposes that $0.999\dots = 1.000$ is a legitimate identity to make [Fowler 1992, 725–733].

disappointed to learn by such “commonplace remark[s] the secret of continuity”, he could not prove the *truth* of his own claims:

I may say that I am glad if every one finds the above principle so obvious and so in harmony with his own idea of a line; for I am utterly unable to adduce any proof of its correctness, nor has any one the power. [Dedekind 1963, 11]

In other words, the assumption of continuity is nothing other than the *axiom* by which mathematicians *attribute* continuity to the line. The notion of “continuity” is an *axiomatic* one. It is not empirical or contingent upon the observation of motion in space. It is an assertion—it is an exercise in logic. Because it is a created concept, Dedekind concluded:

If space has at all a real existence it is *not* necessary for it to be continuous; **many of its properties would remain the same even were it discontinuous**. And if we knew for certain that space was discontinuous there would be nothing to prevent us, in case we so desired, from filling up its gaps, in thought, and thus making it continuous; **this filling up would consist in a creation of new point-individuals and would have to be effected in accordance with the above principle**”. [Dedekind 1963, 12]

In a recent analysis of Dedekind’s account of continuity, Roger Cooke tells us that Dedekind’s aim in linking the definition of continuity to infinitesimal analysis was to address a “foundation problem that had lain beneath the surface of analysis since the time of Descartes” [Cooke 2005, 553]. It was a foundational problem that the ancients had dealt with under the rubric of “incommensurables”—namely, the problem of how to express diagonals and sides of squares that could not be represented as ratios or integers. The problem facing mathematicians right up until Descartes’ time, says Cooke, is that “geometric magnitudes had never been systematized in terms of arithmetic rules, since they had never been thought of as numbers” [Cooke 2005, 554]. As Cooke explains, Descartes was among the first to interpret the product of two lengths as a line as opposed to an *area*. His analytic geometry was not, however, without its problems. The mean-value theorem, for example, relied on some notion of “continuity” to,

Guarantee that a curve containing points on both sides of a line must intersect the line. But, as the Pythagoreans had shown, the numerical version of this theorem was false: the point of intersection might very well not correspond to any number. It was incorrect to call the intersection an irrational *number*, since there was no articulated theory of irrational magnitudes that allowed them to be added or multiplied like numbers. Algebraic rules such

as $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ applied to magnitudes by *fiat*, but were difficult to prove, even geometrically; and no one had produced any corresponding arithmetic rules, or even a non-tautological definition of the square root of a non-integer. Any such definition first of all begged the question of the *existence* of the object defined; and if existence is granted by appeal to geometry, the rules for treating lengths as numbers still needed to be formulated and proved correct. Such was the situation that confronted Dedekind. [Cooke 2005, 555]

Working within this tradition, Dedekind had become concerned over the possibility that continuity was invoked whenever mathematicians were implicitly (or explicitly) referring to ambiguous geometric conceptions (*i.e.*, in their descriptions of the real number line as a geometric magnitude). In line with Riemann and Gauss before him, Dedekind's aim was to prove that new number systems could be constructed, as he did not think they were dependent upon *a priori* physical truths [Cooke 2005, 559]. On Dedekind's view, mathematicians *construct* their tools, including their accounts of continuity, in order to carry on with their research. They adopt such constructed rules as axiomatically true statements about the systems within which they work. In arguing for such an account of mathematics, Dedekind was engaged in the process of separating mathematical constructs from both scientific theorizing and physical existence. "Continuity", he argued, is based on arithmetical rules which involve the notion of a "cut". It exists as a constructed mathematical tool rendered axiomatically true by *fiat*, and used in mathematical puzzle-solving, though not necessarily in scientific theorizing.

3. Clifford's Continuity

Clifford's critical view of continuity underpins the almost virulent empiricism that pervades his entire mathematical project. For Clifford, "continuity" is to be understood as the assumption that there are no physical, non-material gaps in the fabric of space. It is empirically justifiable; it makes sense to believe that space is a continuous fabric of some sort of material ether. It makes sense to believe that space exists everywhere—that there are no gaps of non-existence from this end of the room to the other. It also makes sense to believe that there are no gaps in the continuity of time; there are likely no moments of non-existence that follow moments of existence. In so far as we trust our sensory organs, we are justified, therefore, in believing in the continuity of space and time. Hence, we are justified in basing our mathematical description of physical phenomena upon the assumption that continuity exists as a physical fact.

Clifford's mathematical concerns with continuity cannot be distinguished from his other conceptual commitments, in particular his view on the knowledge of *causes*. Calculus, he argued, involves the art of describing the rate of

change in motion of a point or an object. “Continuity”, upon which this description is based, is fundamentally based upon a justified belief in *spatial and temporal continuity* in the universe. Note that, for Clifford, mathematics is fundamentally empirical, as well as conventional—the former being due to the fact that we can know nothing beyond the provisions of our sensory organs, and the latter being due to the fact that all of our knowledge, no matter how well justified, is ultimately limited by the finite perceptual capacities of our sensory organs. All knowledge is therefore empirically derived and open to future revision. Thus, in considering Clifford’s stand on “continuity” we need to consider two inter-related aspects of his philosophy:

1. Empirical experience (observation) as the basis of *all* mathematical knowledge;
2. “Continuity” as a hypothetical property of space.

Clifford’s acceptance of “continuity” as a spatial property highlights his general philosophy of science, which advocated for an ardent materialism in which cause-and-effect relationships *could only* be accounted for via a series of contiguous particle motions in space. Throughout the 1870s, Clifford felt the need to discuss and explain these beliefs in great detail. His popular lectures and his *Common Sense of the Exact Sciences* (published posthumously in 1885) were aimed at non-specialist audiences for whom the notion of continuity would have been new, or at least unfamiliar, terrain. For instance, when he died, Clifford was engaged in writing a textbook for his first year students at University College London (UCL). In part this was because Clifford was working within a society that favoured and lauded popularizations of science and mathematics. Scientists and popular lecturers such as T.H. Huxley, John Tyndall, and even P.G. Tait (who popularized links between science and religion) had made it the duty of experts to explain their expertise in palatable and popular forms, even when writing for expert audiences. Clifford was hypersensitive to this fact given his employment at UCL, an institution that had been established with the aim of extending the scope of higher education to include students who had emerged from families not traditionally associated with Cambridge or Oxford (*i.e.*, sons of merchants in London, and sons of secular families).⁵

Clifford, however, also devoted a significant amount of space in his expert mathematical papers to discussing the concept. He used those publications to advance controversial claims regarding the foundations of “number” and “quantity”. One such claim was that “number” ought to be understood as *motion*—as a “step in space”—where a “step” could be understood as a simple geometrical displacement. Thus, all number would be geometrical at root, unless specified as “discrete” in nature. “Continuity” would therefore constitute

5. Lightman notes that efforts to popularize science peaked in the late Victorian period, by which point journals and magazines were hiring non-specialists to journalistically present mathematical and scientific discoveries in accessible formats [Lightman 1997].

a fundamental property of “number” understood as “continuous magnitude”. The implication here is that a certain kind of physical space is necessary in order for numbers and their related operations to exist. These specific claims set Clifford apart from the conventionalist philosophy that defined the works of people such as Dedekind, because they indicate Clifford’s deeply empirical approach to mathematical knowledge. His was an approach that accepted the conventional nature of mathematical concepts, but only in so far as those conventions were the product of empirical observations and inferences about the space within which motion is possible.

For Clifford, all we can ever *know* is that which is empirically inferred. It is important, therefore, to qualify Clifford’s view of empirical *inference*. While it is the case that empirical evidence is the best judge of what does and does not exist in this world, *direct* empirical knowledge is limited. It is circumscribed by the finite reach of our sensory capacities. Because of this, Clifford accepted that our mathematical descriptions can only ever be conventional expressions of what we believe to be true about the world. In order to advance mathematical and scientific knowledge, experimentalists and mathematicians had to be willing to speculate upon, and use, concepts related to the nature of the world as they knew it *indirectly*. An example of this approach is found in Sir William Rowan Hamilton’s work on quaternions which, Clifford claimed, demonstrated that complex numbers were not special, mystical, or imaginary “quantities”.⁶ Rather, they could be meaningfully understood when thought of as geometrical operators that caused bodies to rotate in space. Accepting such an interpretation required that mathematicians revise their beliefs regarding the universality of certain mathematical principles, in particular the principle of commutativity. It also required that mathematicians quench any desire to ideologically preserve commutativity in light of the useful tools that could emerge when adherence to such traditional principles were abandoned.

An early indication of what Clifford had in mind by advancing such a philosophy of mathematics is found in his paper, “On the Space Theory of Matter” (delivered to the Cambridge Philosophical Society in 1870 and published in its proceedings in 1876)⁷ [Clifford 1876b]. Although it is unclear when Clifford first read the German version of Riemann’s paper on the “Hypotheses that lie at the base of geometry” (1854), it is likely that upon reading Riemann in the mid-1860s (near the end of his university career), he began to further develop his view of mathematical knowledge as an empirical craft. It is also at that time that he likely began to develop his specific views regarding the physical nature of mathematical “continuity”. In his 1870 presentation, Clifford posited

6. See [Pycior 1997] and [Pycior 1976] for discussions of the mysterious nature accredited to imaginary numbers well into the 19th-century.

7. The Cambridge Philosophical Society was set up in 1819 to promote “scientific inquiry” and facilitate “the communication of facts connected with the advancement of Philosophy and Natural History”. We can consider this paper as one delivered to an audience composed of experts, and students on their way to becoming mathematical experts.

that continuity was a workable hypothesis, which enjoined scientists and mathematicians to adhere to a fundamentally *material* view of the universe such that the interaction between physical entities (say, of atoms at the most basic level of material reality) would dominate descriptions of physical phenomena. Clifford wrote that if scientists were to adopt the Riemannian hypothesis that space might not be flat (*i.e.*, that it might possess an inherent curvature) then the continuity hypothesis would lead to the view that non-flat geometrical structures could *cause* physical phenomena. In other words, continuous geometrical displacements in non-flat space might manifest themselves in the form of seemingly discontinuous phenomena perceived by humans in Euclidean space (*i.e.*, as gravity, electromagnetism, or colour change) [Clifford 1876b, 21].

Clifford's justification for advancing such a hypothesis was in part physiological. A scientist's sensory perceptions are limited, he wrote. Consider the fact that the axioms of plane geometry work on a piece of paper, while in reality, we know the piece of paper is covered with "small ridges and furrows, upon which (the total curvature not being zero) these axioms are not true" [Clifford 1876b, 21]. The scientist's limited experience of space may suggest that the axioms of solid geometry are more or less true for finite portions of space, but we "have no reason to conclude that they are true for very small portions, and if any help can be got thereby for the explanation of physical phenomena, we may have reason to conclude that they are not true for very small portions of space" [Clifford 1876b, 21]. Clifford combined his views of geometry with his overarching belief in the existence of spatial and temporal continuity in order to make the following claims:

- (1) That small portions of space *are* in fact of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them;
- (2) That this property of being curved, or distorted, is continually being passed on from one portion of space to another after the manner of a wave;
- (3) That this variation of the curvature of space is what really happens in that phenomenon which we call the *motion of matter*, whether ponderable or etherial;
- (4) That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity [Clifford 1876b, 21–22].

This paper stands as one of many in which Clifford built up a new philosophical approach to mathematical and scientific theorizing. In Clifford's worldview, if mathematicians and theorists were to consistently abide by the physical principle of "continuity", explanations of seemingly discontinuous events (such as gravitational attraction between bodies) could lead to new conceptualizations of spatial structure (*i.e.*, non-flat spaces).

4. Clifford the Riemannian

Based on his view of continuity, Clifford developed operational tools which he called bi-quaternions. In so doing, he aimed to expand upon both Hamiltonian quaternions as well as Grassmannian algebra.⁸ Clifford's bi-quaternions constituted another form of non-commutative algebra involving operators that could be used to map the motion of rigid bodies in Euclidean and non-Euclidean spaces. Throughout the 1870s, Clifford developed his bi-quaternions in the hopes of linking the seemingly empirical reality of Euclidean space to a geometric model that would be able to describe rigid body motion in non-flat spaces. Lauded in a posthumous review (by his biographer H. J. Stephen Smith), Clifford's contributions on bi-quaternions included: "Preliminary Sketch of Bi-quaternions" [1873], "A Further Note on Bi-quaternions"⁹ [1876], and his book-length account *Elements of Dynamic: an introduction to the study of motion and rest in solid and fluid bodies* [1878]. In those works, Clifford outlined his development of various new mathematical operators called "rotors", "motors" and "bi-quaternions", which resulted from his continuing interest in the description of physical motion in Riemannian manifold space. As Smith wrote,

Riemann's ideas of a constant curvature in space lies at the bottom of Clifford's theory of bi-quaternions, to which he devoted much continuous thought, and which was the origin of his researches into the classification of geometric algebras.¹⁰ [Tucker 1882, xlv]

Indeed, Clifford's work constituted a major break with traditional geometrical thinking in that:

Some men who have an ardent love for new knowledge find it difficult to maintain an unflagging interest in geometry, because they regard it as a purely deductive science of which the first principles (axioms, postulates, and definitions), whether derived from experience or not, are unquestionable and contain implicitly in themselves all possible propositions concerning space. Thus, the unknown, or at least the unforeseen, seems to be excluded from geometry, because whatever may be found out hereafter must be latent in what is already known". [Tucker 1882, xxxl]

Such staid attitudes had begun to change in Great Britain in the 1860s, and they became more prominent throughout the 1870s, largely due to the popular-

8. Only some of the Hamiltonian aspects of Clifford's work will be highlighted here; I elaborate upon the Grassmannian aspects of his work in my PhD thesis (to be submitted June 2009).

9. These two papers were left untitled by Clifford; their respective names are to be credited to Robert Tucker.

10. According to Tucker, nine-tenths of the papers in his *Mathematical Papers* are "geometrical" in nature.

izing efforts by Hermann von Helmholtz and Clifford. Those two mathematicians defended the usefulness and justifiability of non-Euclidean geometrical models in mathematical physics. As Tucker recounted:

Upon the view put forward by Riemann and adopted by Clifford, the essential properties of space have to be regarded as things still unknown, which we may one day hope to find out by closer observation and more patient reflection, and not as axioms to be accepted on the authority of universal experience, or of the inner consciousness. [Tucker 1882, xl]

Both Clifford and Helmholtz emphasized the conventional and hypothetical nature of the Euclidean axioms and the consequent need for an empirically-based revision of Euclidean-dominated mathematics and research programmes found within the university settings of Cambridge, UCL, and other British educational institutions. Clifford's concerns regarding "continuity" formed part of this wider empiricist agenda, which included the introduction of non-Euclidean geometries to mathematical physics and the reform of mathematics education in general. Clifford argued that bi-quaternions were the result of observing "continuity" in physical processes, which he felt required more powerful conventional description than that provided by the Cartesian coordinate system.

Tucker recounts four assumptions Clifford made regarding the underlying "ordinary conceptions about space". Those "conceptions" included the following:

- Spatial continuity;
- Spatial flatness at microscopic levels;
- Spatial similarity at every point;
- The possibility of the existence of figures similar to one another, but on different scales of magnitude [Tucker 1882, xli].

As mentioned earlier with regards to "continuity", Clifford was philosophically ready,

To adopt either of the two opposite hypotheses that space is continuous or that it is discontinuous, while admitting fully that no phenomena have yet been observed which point to its discontinuity. [Tucker 1882, xli]

In his research, Clifford found it productive to assume that continuity did exist, while the other axiomatic claims in mathematics could be revised around it.

With regards to the second assumption, Clifford followed Riemann in arguing that mathematicians and scientists had to be willing to reject the notion that between every three points very near to one another there always exists a triangle with an internal angle value of 180 degrees. The idea that objects

may not stay the same shape as they move through space suggests that space might not be homogeneous. Clifford thought the topological structure of space might vary from one section of space to another. Thus, our perceptions of an object in one section of space compared to our perceptions of that same object in another section of space might yield very different empirical results. These views underpinned Clifford's belief that action-at-a-distance "forces" (such as Newtonian gravity) were describable in terms of simple movements (or displacements) of bodies in non-flat or elliptic space. Indeed, for Clifford, any empirical perceptions of inexplicable, mysterious, or mystical "forces" were an indication of a non-Euclidean ontological curvature requiring a new form of geometrical description.

One of the seminal influences that helped to shape Clifford's views on these matters can be found in Riemann's paper, "On the Hypotheses which Lie at the Bases of Geometry" (originally delivered in 1854). In that paper, Riemann argued that "geometry assumes, as things given, both the notion of space and the first principles of constructions in space" [Riemann 1867, 55]. According to Riemann, geometry contains definitions of these assumptions; yet, those definitions remain merely "nominal":

The true definitions appear in the forms of axioms. The relation of these assumptions remains consequently in darkness. We neither perceive whether and how far their connection is necessary, nor, *a priori*, whether it is possible. [Riemann 1867, 55]

Riemann set as his own task that of clarifying those nominal relationships by reconsidering the idea of magnitudes, in particular "extended magnitudes, including space" [Riemann 1867, 55]. He aimed to construct "the notion of a multiply extended magnitude out of general notions of magnitude", the upshot of which is the fact that multiply extended magnitudes require differing measure-relations [Riemann 1867, 55]. Different spatial structures require different metric systems.

One consequence of Riemann's reconsideration of measure relations in variously extended magnitudes is that "the propositions of geometry cannot be derived from general notions of magnitude, but (...) the properties which distinguish space from other conceivable triply extended magnitudes are only to be deduced from experience" [Riemann 1867, 56]. Triply extended magnitudes (space of three dimensions) that can be measured using Euclidean lines and line segments on a Cartesian graph, constitute only one particular case [Riemann 1867, 56]. That is to say the system Euclid had established constitutes only one system of measure-relations in space. Many other systems could, in theory, be created.

In Riemann's opinion, the Euclidean construction should therefore be considered to be an hypothesis: "These matters of fact are, like all matters of fact, not necessary, but only of empirical certainty; they are hypotheses" [Riemann 1867, 56]. Scientists could investigate the "*probability*" that humans live within

a spatial fabric that abides by Euclidean measure relations, but the establishment of such probabilities would be dependent upon the powers of empirical observation open to us. Therefore, it remains incumbent upon mathematicians to always question the validity or “justice” of “their extension beyond the limits of observation” [Riemann 1867, 56]. Riemann, therefore, suggested that mathematicians extend their systems to include models that could describe and grasp that which is not directly observable.

In clarifying his notion of an n -ply extended magnitude, or “manifold”, Riemann considered two types of manifolds: *continuous* and *discrete*. This distinction is one that Clifford would later reproduce in his own mathematical papers and in the *Common Sense of the Exact Sciences*. Examples of discrete manifolds, Riemann wrote, are common in everyday language. Any group of distinct things, such as the letters in the alphabet, can be considered to be a discrete manifold. Continuous manifolds, on the other hand, are more difficult to imagine. One common example is the position of perceived objects, when definite portions of manifoldness are distinguished by boundaries, which themselves take up no space¹¹ [Riemann 1867, 57].

In addition, the difference between discrete and continuous “quantity” can be made clear by discussing discrete number in terms of “counting” and continuous magnitude in terms of “measuring”. Riemann wrote that “measure consists in the superposition of the magnitudes to be compared. It therefore requires a means of using one magnitude as the standard for another” [Riemann 1867, 57]. In the absence of a “standard”, two magnitudes can only be compared “when one is a part of the other, in which case we can only determine the more or less and not the how much” [Riemann 1867, 57]. The question of “How much?” is, therefore, only applicable in a continuous manifold when discussing “*How many units of measure?*” and not “*How much of the thing itself?*” since a continuous system is not open to discrete divisions¹² [Riemann 1867, 57].

“Continuity” serves as the basis of continuous manifolds, as it implies a certain kind of spatial structure—one in which a point or object can move about in a variety of directions without interruption. The number of directions of movement determines the type of space being defined. “If one regards the variable object instead of the determinable notion of it,” Riemann wrote, the construction may be described as a “composition of a variability of n dimensions and a variability of one dimension” [Riemann 1867, 58]. Riemann argued that in any given manifoldness (any n -ply extended space) there are n determinations

11. Riemann called these portions “quanta”.

12. Riemann noted that it was this distinction that formed “a general division of the science of magnitude in which magnitudes are regarded not as existing independently of position, and not as expressible in terms of a unit, but as regions in a manifoldness”. By the time Riemann’s paper was published in 1854, such research had become a necessity. The previous lack of such considerations was “no doubt a chief cause why the celebrated theorem of Abel, and the achievements of Lagrange, Pfaff, Jacobi for the *general theory of differential equations*, have so long remained unfruitful” [Riemann 1867, 57].

of quantity which, together, define the position of the object (for example, in Cartesian space, x -, y -, and z -coordinates provide three “quantities” that together give us the precise location of an object). In any manifold space, “The determination of position in it may be reduced to n determinations of magnitude” [Riemann 1867, 59]. Measure-relations can be studied in the abstract, because their dependence upon one another can be represented by formulae. However, geometric representations always lay at the bases of these formulae.

In sum, Riemann’s working hypothesis was that *position*-fixing can be reduced to *quantity*-fixing in an n -dimensioned manifoldness. If the *position* of an object changes, its displacement can be represented by a relative change in *quantities* (measure-relations) in a given number of directional dimensions. In cases in which the manifold contains no curvature, the equivalent mathematical model at play is a Euclidean one. However, Riemann contended that such a model should be considered to be the *exception* and not the norm, because in such spaces figures could be moved about with no change in their size or shape.¹³

In the final section of his 1854 paper, Riemann elaborated upon the physical implications of his research by arguing that what remained to be done was to determine “the degree to which these assumptions are borne out by experience” [Riemann 1867, 67]. Indeed, “It is upon the exactness with which we follow phenomena into the infinitely small that our knowledge of their causal relations essentially depends,” he wrote, adding that it is possible, and even likely, that conventional Euclidean measurements based upon “the notion of a solid body and of a ray of light, cease to be valid for the infinitely small.” Instead, the only believable assumption that might remain is one of spatial continuity—all else needs to flow from this conception of the universe, thus leading to the rejection of action-at-a-distance principles, and other “discontinuities” prominent in scientific theorizing [Riemann 1867, 68]. The entire structure of good science, therefore, depends upon recognition of this fact. Riemann concluded:

Researches starting from general notions, like the investigation we have just made, can only be useful in preventing this work from being hampered by too narrow views, and progress in knowledge of the interdependence of things from being checked by traditional prejudices. [Riemann 1867, 69]

Thus, the mathematician must be willing to engage in another domain of knowledge, “another science” altogether, in order to determine and devise good mathematics—namely, physics.

13. This claim was fundamental for both Helmholtz and Clifford’s respective philosophies of science, in which they each sought to defend the view that non-Euclidean geometry is physiologically justifiable because, even in uniformly curved space, we do not *perceive* any change in shape of a body as it moves (even though the underlying spatial structure might be non-flat).

In a recent account of Riemann's seminal paper, Jeremy Gray argues that Riemann's lecture (published posthumously in German in 1867) was to prove "central to the overthrow of Euclidean geometry as the source of geometrical ideas" [Gray 2005, 506]. The idea of a multiply extended magnitude was, roughly, something that could be measured by a given number of coordinates. Riemann concluded that geometry could be thought of as dealing with various spaces (n -fold extended magnitudes) in which distances were to be measured using an infinitesimal ruler [Gray 2005, 510]. If the infinitesimal measuring rod could be placed anywhere and still offer the same scale of measurement, it would mean that the space being measured is constant in curvature. Thus, the sum of the angles in any one triangle in such a spatial model would be equivalent to the sum of triangles at all places. However, Riemann ultimately wanted to discuss the measurement of distances between points without having to rely on Euclidean space.¹⁴ In addition, at the time of Riemann's *Habilitation*, Gray writes, he was working in Wilhelm Weber's physical laboratory. Riemann had therefore been occupied with issues of gravitation, electricity, and magnetism—all of which had the property of acting across vast distances at "enormous speeds" [Gray 2005, 513]. Gray thus accounts for Riemann's approach to mathematics in the following way:

[Riemann] apparently sought to explain this by imagining that the fabric of space was in some way subtly altered, and distortions in it spread like ripples. He was not able to work this up into a coherent theory, but it seems clear that this idea of conceptually rethinking the nature of space in a direct physical context accounts for many features of his lecture. It underlines the avowedly empirical nature of the lecture, it is in line with Riemann's attempt to make proposals that allow one to explain phenomena, and it explains the somewhat anti-Newtonian rhetoric. [Gray 2005, 513]

Clifford adopted all of these Riemannian themes. He assumed that space is not discontinuous. Rather, he argued, the principle of "continuity" prevails and mathematical-physical representations of dynamics had to adapt to that fact. Spatial "continuity" allowed for the transmission of action between cause and effect. Clifford's repeated attacks on gravity—a force that he underlined as problematical—drew heavily from Riemann's own attacks on such action-at-a-distance concepts. Clifford's idiosyncratic approach to these topics, however, comes from his engagement with both Hamiltonian quaternion analysis and Grassmannian algebra, both of which aided in his development of bi-quaternions. Clifford's bi-quaternions kinematically describe the motion of rotating rigid bodies. They offer the possibility of describing physical phenomena in continuous, non-flat space—this being an extension, in Clifford's opinion, of Riemann's own work.

14. Gray writes that Riemann's "inspiration here was Gauss's discovery of curvature, which Gauss showed was something that could be determined from quantities measured in the surface alone" [Gray 2005, 509].

5. A Further Note on Bi-Quaternions

Clifford built upon Riemann's distinction between discrete and continuous manifolds in his "Further Note on Bi-quaternions" by highlighting the difference between discrete numbers and continuous "steps in space". The notion of a "step in space" constituted a unifying theme throughout Clifford's works, in which he described arithmetic, algebra, and calculus in terms of simple "steps in space". For the purposes of this paper, I will focus on Clifford's brief discussion of the theme in the "Further Note on Bi-quaternions". This paper demonstrates Clifford's view that mathematical concepts do reflect the "continuity" of physical space and, thus, they can be described as "steps in space".

Recall that, for Clifford, "continuity" is ultimately an *assumption*. Mathematical knowledge remains a conventional language open to indefinite revision limited only by that which is empirically possible. Consider the following equation:

$$2 \times 3 = 6$$

Following in the tradition of the Newtonian polar coordinates, which posited that a set of polar coordinates (r, θ) does not define a point uniquely, Clifford suggested that an arithmetic operation is alone open to multiple representations. In the equation above, two interpretations exist. In the first instance, "3" can be considered a "concrete number of things" in which "2" is the operation of doubling such that we would read the entire equation as "doubling three marbles makes six marbles" [Clifford 1876b, 385]. In the second instance, both "2" and "3" can be abstract numbers which affirm the existence of a third number "6". That third number has a definite relation to the other two numbers only as their product (*i.e.*, as the result of an *operation*, rather than as a quantity on its own) [Clifford 1876a, 385]. Clifford contended that "various meanings" could be given to the numbers "2" and "3":

We have regarded 2 as a symbol of operation, 3 as a concrete number, and 6 as a concrete number. But we may also regard all three symbols as symbols of operation, and so read the formula 'doubling the triple of anything makes the sextuple of it'. [Clifford 1876a, 385]

Similarly, the equation $abc = d$ possesses two interpretations, Clifford wrote:

- 1) a times b times c things makes d things;
- 2) a times b times c times anything makes d times that thing [Clifford 1876a, 386].

The last symbol can be regarded either as a concrete *number* or as a symbol of *operation*. Such varying interpretations allow one to easily extend the concepts of addition, subtraction, and other arithmetical processes to "steps", Clifford contended. For example, he provided a double meaning to the symbols "+"

and “-”. In the first instance, the symbols would indicate the *direction* of a “step”. The symbol (+3) would therefore indicate a step of 3 units *forward*, while the symbol (-3) would indicate a step of three units *backwards*. “When these symbols are attached to an operation performed upon steps, they mean *retaining* and *reversing* respectively,” Clifford explained [Clifford 1876a, 386]. Thus, the equation $(-2)(+3) = (-6)$ would be open to two interpretations:

- 1) Doubling a step of three forward and reversing it makes a step of 6 backward;
- 2) [Or] To triple a step and retain its direction then to double and reverse it is the same as to sextuple and reverse it [Clifford 1876a, 387].

Given the controversy surrounding “quaternions” as initially proposed by Sir William Rowan Hamilton in the first half of the 19th century (Pycior 1976), and given that the debate regarding the foundations of mathematical “number” (in particular, negative numbers and the square roots of negative numbers) had not entirely died down by the time Clifford chose to develop and extend Hamiltonian quaternions and Grassmannian algebra, it is clear that this seemingly basic introduction to the interpretability of mathematical concepts in the “Further Note” in fact constituted an important philosophical statement on Clifford’s part. Clifford’s aim was to demonstrate that since simple mathematical concepts such as addition and subtraction could be treated in an equivocal and malleable way, so too could more complex concepts such as Hamiltonian vectors. The “rules” of arithmetic were by no means an ideological barrier to the establishment of new mathematical system, especially if the system in question upheld the principle of geometrical and physical continuity (as we will see). In Clifford’s opinion—and against many of Hamilton’s critics—quaternion algebra did not necessitate any philosophical conundrums or lead to any troubling debates over the *a priori* status of mathematical knowledge. It was as simple a step in logic for an empiricist such as Clifford to afford a dual interpretation to the new system of non-commutative algebra.

In order to see how Clifford further extended Hamiltonian quaternions on the basis of “continuity” as a physical fact, it is useful to provide a brief account of what it is that Hamilton was actually claiming. It was during his time as a student at Trinity College, Dublin, from 1823 to 1827, that Hamilton had been exposed to the French works of P.S. Laplace, J.L. Lagrange, S.D. Poisson and S.F. Lacroix [Lewis 2005, 460]. Some of those influences led him to consider a problem that many mathematicians in the 1830s had considered to be one of the most important of the day: how to extend the system of number pairs (which represented complex numbers), to triples of numbers such that the same operational properties were preserved [Lewis 2005, 461]. In his account of Hamilton’s efforts to do so, Lewis summarizes the problem in the following way:

A complex number $z = a + bi$ can be represented in the Euclidean plane by the directed line segment from the origin to the

point with real number coordinates (a, b) . On multiplying z by $i (= \sqrt{-1})$, the result would be the segment from the origin to $(-b, a)$ which can be regarded as the result of rotating the original segment 90° counter-clockwise about the origin. The problem could thus be put in a geometrical and somewhat more general fashion: How can this mathematical operation, represented by rotation about a point in the plane, be extended to rotation about a line in three dimensions? Expressed this way the answer turns out to be that four numbers, not three, are required. [Lewis 2005, 461]

Hamilton's construction was to introduce the now well-known imaginary entities, i, j, k , which abided by the following rules:

$$i^2 = j^2 = k^2 = -1$$

$$ij = k$$

$$jk = i$$

and,

$$ijk = -1$$

The resulting “hypercomplex number”—or Hamilton's quaternion (q)—was formed as:

$$a + bi + cj + dk,$$

where a, b , and c are the real number components of the expression. Hamilton made the following claims in regards to the expressions $\beta \div \alpha = q$ and $q \times \alpha = \beta$:

The quotient q can be regarded as an operator that produces one directed line segment from another. If q is a ‘tensor’, or signless number, then it affects only the length of α . If q is a sign (+ or −) it changes the direction of α . If it is a real number then q may have the effect of changing both the direction and length. If it is a ‘vector-unit’ (or ‘quadrantal vector’), i, j, k , then the effect is to turn α right-handedly through 90 degrees in a plane perpendicular to the vector-unit. Hamilton points out that a multiplication of a vector-unit, say i , by itself results in a rotation of 180° , *i.e.*, the same as multiplying by -1 (reversing its direction) or $i^2 = -1$. [Lewis 2005, 464]

Hamilton had called the addition of a scalar quantity and a vector a “quaternion”. It was an entity that contained both a scalar part and a vector part and caused rotation about an axis with a magnitude change in a given vector.

Clifford repeated much of Hamilton's approach in introducing his own bi-quaternions to readers in the 1870s. If mathematicians assumed that the law of addition of vectors held in a plane, such that $AB + BC = AC$, then they could

FIGURE 1: [Clifford 1882, 414]

interpret the “so-called *imaginary* or *impossible* quantities [as] the operators which convert one vector into another,” Clifford wrote [Clifford 1876a, 386] (See Figure 1).

On this interpretation, i operates on a vector causing it to turn counter-clockwise through a right angle so that:

$$i \times OE = OA.$$

If $a = OM/OA$ and $b = MB/OA$, where a and b are ratios of vectors in a line (as defined above), then Clifford argued that every expression of the form $a + bi$ is a ratio of two vectors [Clifford 1876a, 387]. By giving “proper values” to a and b every *vector* in the plane can be represented by $(a \times OA) + (b \times OB')$ [Clifford 1876a, 387] such that:

$$OA = j;$$

$$OA' = k;$$

then

$$ij = k;$$

$$ik = -j.$$

Turning to the problem of vectors in *space*, Clifford noted that “the operation which makes one vector into another is of the form $a + bq$ ”, where Q turns through a right angle in the plane of the two vectors.

Note that Clifford’s idiosyncratic approach to this notion was to search for a mechanism by which vectors were not limited to turning through right angles in their plane of existence. Rather, he wanted to develop a mechanism by which one could transform a vector in a given plane into a vector in an alternative plane. To do so, he had to extend the notion of vectors as directed

lines (or steps) in space to directed, positioned, *and* rotating lines (or steps) in space. The emphasis on *position* in Clifford's bi-quaternions is largely a product of his engagement with Riemannian geometry, as explained earlier. Riemann had emphasized the crucial aspect of position in space, given that the same body in two different locations of space might behave differently depending on the geometrical curvature of that particular segment of space (thus, what we perceive of as being discontinuous “forces” acting on bodies is explicable by reference to continuous motion in non-flat spaces). Thus, in Clifford's extension of Hamiltonian quaternions, the principle of continuity remains intact—bi-quaternions operate on entities that are composed of both rotors and vectors (namely, “motors”) in continuous physical space. If space were discontinuous, the mathematical use of these operators to describe physical phenomena would not be possible, because of spatial discontinuity, would render biquaternion operations meaningless.

Thus, Clifford clarified the construction of a bi-quaternion in the following way. First, “the velocity of a rigid body may be represented in one way only as a *rotation-velocity* (ω) about a certain axis combined with a *translation-velocity* (v) along that axis” [Clifford 1873, 182]. This combined velocity is what he called a *twist-velocity* about a particular screw, where the axis of the screw was the axis of rotation, and its pitch was the ratio v/ω (the ratio of the translation to the rotation). Clifford then suggested the following terms:

1. A “screw” is the geometrical form resulting from the combination of an *axis*, or straight line, with a *pitch*, which is a linear magnitude;
2. A “wrench” is the association of a screw with a magnitude “whose dimensions are those of a *force*”;
3. Finally, a “*twist-velocity*” is the association of a magnitude whose dimensions are those of an angular velocity [Clifford 1873, 183].

Along with these new basic spatial-geometrical entities, Clifford defined his more complex creations such as “motors” in the following way:

Just as a vector (translation-velocity, or couple) is magnitude associated with direction, and as a rotor (rotation-velocity, or force) is magnitude associated with an axis, so this new quantity, which is the sum of two or more rotors (twist-velocity or wrench) is magnitude associated with a screw. [Clifford 1873, 183]

The simplest example of a “motor” involved the general motion of a rigid body.

With these new tools in hand, and assuming *continuity* to hold as a spatial fact, Clifford explained the operation of quaternions and bi-quaternions as follows. If we have the vectors AB and AC , both of which start from any arbitrary point A , we can make AB identical to AC by “turning it round an axis through A perpendicular to the plane BAC until its direction coincides

FIGURE 2: [Clifford 1882, 228]

with that of AC , and then magnifying or diminishing it until it is of the same length as AC " [Clifford 1873, 183] (see Figure 2).

The ratio of the two vectors is the combination of an ordinary numerical ratio (magnitude change) with a rotation (direction change). In Hamilton's terms, this means that a quaternion is the product of a tensor and a versor. Since point A is perfectly arbitrary, "this rotation is not about a definite axis, but is completely specified when its angular magnitude and the direction of its axis are given" [Clifford 1873, 184]. The mathematician can therefore write that:

$$AC/AB = q [\text{quaternion}].$$

For Clifford, this is an operation which,

Being performed on AB , converts it into AC , so that $q \cdot AB = AC$. The axis of the quaternion is perpendicular to the plane BAC ; and it is clear that the quaternion operating upon any other vector AD in this plane will convert it into a fourth vector AE in the same plane, the angle DAE being equal to BAC and the lengths of the four lines proportionals. [Clifford 1873, 184]

The problem, as mentioned earlier, was that a quaternion could only operate upon a vector that was perpendicular to its axis: "If AF be any vector not in the plane BAC , the expression $q \cdot AF$ is absolutely unmeaning," Clifford wrote [Clifford 1873, 184]. He, therefore, set as his task the construction of an operator by which one could convert a *rotor* into another *rotor*. In so doing he found that he had to (see Figure 3):

Turn A about the axis AC into the position AB' , parallel to CD . Then slide it along this axis into the position CD' . Lastly, magnify or diminish it in the ratio of CD' to CD . The first two operations may be regarded as together forming a twist about a screw whose axis is AC and whose pitch is $\frac{AC}{\text{circ. meas. of } ABB'}$. [Clifford 1873, 184]

Clifford explained that the ratio of two rotors constitutes the combination of an ordinary numerical ratio with a twist, where a twist indicates a definite "screw" and is "only specified when its angular magnitude and the screw

FIGURE 3: [Clifford 1882, 228]

(involving direction, position, and pitch) [are] given” [Clifford 1873, 185]. He then made the following analogy: “Just as the rotation (versor) involved in a quaternion is the ratio of two directions, so the twist involved in the ratio of two rotors is really the ratio of their axes”. Thus, by

Using the expression *tensor-twist* to mean *the ratio of two rotors* (which is in fact a twist multiplied by a tensor), we may say that a tensor-twist can operate upon any rotor which meets its axis at right angles. [Clifford 1873, 185]

Like the quaternion, Clifford’s ratio of two rotors also had a “restricted range of operation” [Clifford 1873, 185]. The question Clifford needed to answer was: “What is the operation that converts one motor [which performs a screw operation] into another?” [Clifford 1873, 185]. When two motors have the same pitch, he said, the answer is easy: their ratio is a tensor-twist. The tensor is the ratio of their magnitudes, and the twist is the ratio of their axes.¹⁵ The more general case, however, in which the pitches of two motors are different, is not so simple to solve. If a motor consists of a *rotor part* and a *vector part*, then its pitch is determined by the ratio of these two parts. Clifford determined this ratio by combining a “suitable vector with a motor” such that the pitch could be made to be “anything we like, without altering the rotor part” [Clifford 1873, 186].

15. We are led to this conclusion, he wrote, “by considering each motor as the sum of two rotors which do not intersect” [Clifford 1873, 186].

Geometrical Form	Quantity	Example	Ratio
<i>Sense</i> on a straight line	<i>Vector</i> on a straight line	Addition or Subtraction	Signed Ratio (+, or -)
<i>Direction</i> in Plane	<i>Vector</i> in Plane	Complex quantity	Complex ratio [where $\sqrt{-1}$ is an operator that turns vectors]
<i>Direction</i> in Space	<i>Vector</i> in Space	Translation, Couple	Quaternion
<i>Axis</i>	<i>Rotor</i>	Rotation-Velocity, Force	Twist
<i>Screw</i>	<i>Motor</i>	Twist-Velocity, System of Forces	Biquaternion

Therefore, to determine the operation that could convert motor A into motor B , Clifford first let B' be a motor that had the same rotor part as B and the same pitch as A . To convert motor A into vector β (where β is a vector parallel to the axis of B), he introduced the symbol (ω) “whose nature and operation will at first sight appear completely arbitrary, but will be justified in the sequel.” Analogously to quaternions, “the symbol ω , applied to any motor, changes it into a vector parallel to its axis and proportional to the rotor part of it” [Clifford 1873, 186]. This operation (ω) changes a rotation about an axis into a translation parallel to that axis. The ratio of two motors can then be expressed as the sum of two parts—one of which is a tensor-twist and the other of which is ω multiplied by a quaternion [Clifford 1873, 188]. This ratio is Clifford’s biquaternion.

Clifford concludes by offering the following table of concepts to his readers to clarify the relationships that he has created between his various new entities in the extension of quaternion mathematics—entities that operate upon physical bodies in continuous space.

6. Specialist and Non-Specialist Statements

It is possible now to paint the picture of a mathematician who could have chosen to pursue research based on discontinuous spatial models (such as action-at-a-distance theories), but who chose, not to do so. Rather, Clifford pursued research based on models of elliptic and non-flat spaces, founding his pursuits on the assumption that continuity existed as a structural aspect of space. His aim was to develop tools that would help to describe rigid body motion by extending what he thought was a useful starting concept (namely, Hamiltonian quaternions). The importance of “continuity” as the underlying basis for Clifford’s advanced mathematical constructions is highlighted clearly in two of Clifford’s early lectures on related topics—the first being “On the theories of the physical forces” (delivered in 1870 to the Royal Institution), and the second being “Atoms” (delivered in 1872 to a popular audience at the Sunday Lecture Society).

“On the theories of the physical forces” describes what Clifford deemed to be the three historical stages of scientific belief, as well as his corollary belief in the need to encourage a new “fourth” stage of scientific thought. The first stage, Clifford argued, had involved the basic observation of phenomena such as planetary motions; the second stage had involved theological reasoning as to the causes of those phenomena. In the third stage—the “era of modern science”—partially materialist conceptions of cause (such as “force”) had come into play. However, those conceptions often relied upon metaphysical assumptions. In the new fourth stage of science—the stage that Clifford felt himself to be engaged in as a reformist Victorian mathematician—the conception of “force” would disappear altogether. Clifford’s aim in developing bi-quaternions was to encourage a movement towards that “fourth” stage, by designing mathematical tools that would precipitate the transition away from action-at-a-distance principles (and their inherent discontinuities) and towards new theories of continuous, non-flat spatial ontology and geometry.

In order to achieve these goals scientists had to confront the underlying question: “What is it that lies at the bottom of thing?” In Clifford’s view, this question is composed of two distinct queries:

- 1) *Why* do things happen?
- 2) *What* is it precisely that does happen? [Clifford 1870, 110].

The first question, he stated, is external to the province of science, Clifford argued. It is a question whose answer lay outside the domain of human knowledge.¹⁶ For the scientific inquirer, there is no hope of answering the question “Why?”, unless an appeal to speculative metaphysics comes into play. The second question is one that Science can aim to answer. But in order for science to supersede the enquiry into *why* things happen, and to advance into the domain of *what* is happening, Clifford argued that scientists must rely upon the “hypothesis of continuity”. This is because the “hypothesis of continuity involves such an interdependence of the facts of the universe as forbids us to speak of one fact or set of facts as the *cause* of another set of facts” [Clifford 1870, 111].

In his popular account of continuity, Clifford wrote:

16. For an account of Clifford’s refusal to consider the “Why?” in scientific theorizing, consider the following polemical statement: “Reflect on the fact that for a single particle, quite irrespective of everything else—the history of eternity is contained in every second of time, and then try if you can find room in this one stifling eternal fact for any secondary causes and the question why? (...) Why does the moon go around the Earth? When the solar system was nebulous, anybody who knew all about some one particle of nebulous vapour might have predicted that it would at this moment form part of the moon’s mass, and be rotating about the earth exactly as it does. But why with an acceleration inversely as the square of the distance? There is no why (...) The cause is only the fact that at some moment the thing is so—or rather, the facts of one time are not the cause of the facts of another, but the facts of all time are included in one statement, and rigorously bound up together” [Clifford 1870, 123].

Things frequently move. Some things move faster than others. Even the same thing moves faster at one time than it does at another time. When you say that you are walking four miles an hour, you do not mean that you actually walk exactly four miles in any particular hour; you mean that if anybody did walk for an hour, keeping all the time exactly at the rate at which you were walking, he would in that hour walk four miles. But now suppose that you start walking four miles an hour, and gradually quicken your pace, until you are walking six miles an hour. Then this question may be asked: Suppose that anybody chose a particular number between four and six, say four and five-eighths, is it perfectly certain that at some instant or other during that interval you were walking at the rate of four miles and five-eighths in the hour? Or, to put it more accurately, suppose that we have a vessel containing four pints of water exactly, and that somebody adds to it a casual quantity of water less than two pints. Then is it perfectly certain that between these two times, when you were walking at four miles an hour, and when you were walking six miles an hour, there was some particular instant at which you were walking exactly as many miles and fractions of a mile an hour as there are pints and fractions of a pint of water in the vessel? The hypothesis of continuity says that the answer to this question is yes: and this is the answer which everybody gives nowadays; which everybody has given mostly since the invention of the differential calculus. [Clifford 1870, 112]

This conclusion is a matter of “fact”, and not “calculation,” Clifford contended. In other words, the hypothesis of continuity is a hypothesis about the nature of matter and the nature of motion; it is a statement about the ontology of the universe. In Clifford’s view, the dependence of calculus upon “continuity” indicates that calculus describes motion in space as it exists. Calculus, therefore, is not an abstract mathematical language disjointed from empirical observation, as Dedekind and his supporters would have argued. The techniques of calculus involve measurements of space over time. The mathematical abstraction of these techniques into symbolic language does not sever their meaning from those empirical and physical roots.

While it might seem obvious to the modern reader that calculus involves an empirical measure of objects in motion, Clifford’s audiences would not necessarily have considered mathematics, in general, to be a fundamentally empirical field of knowledge. The vehemence with which Clifford emphasized the empirical basis of continuity indicates the ideological nature of his statements. In taking such a stand, Clifford was replying to members of the Analytic Society of the early 1820s, and their later followers at Cambridge, as well as formalized approaches to continuity such as Dedekind’s, which sought to establish meanings for mathematical symbols that were internally defined (*i.e.*,

not linked to external phenomena). For instance, in a subtle criticism of such thinkers, Clifford invoked the physical image of “continuity” by describing two rows of points that represent a series of positions in space. The lower row of points represents a series of instants in time in which it is conceivable that some thing might exist. At the instant of time represented by the first point in the lower row, the object in question holds a position in space represented by the first point in the upper row. That object only exists there for an instant, and then it disappears, such that at the “succeeding instants where the lower points have no points directly above them, the thing is nowhere at all” [Clifford 1870, 116]. At the instant of time when there is a space-dot above the object, the thing exists in that space-position. A motion is discontinuous, therefore, when the thing is in “different places at different times, though it is not at all times that it exists at all.” In a discontinuous world, in other words, an object appears and disappears between points in space. When it disappears, it has no existence, which means that the object goes through a series of creations, destructions, and re-creations as time moves forward. Mathematically, such discontinuities arise when “the thing passes from one position to another *distant* from it without going through any intermediate position” [Clifford 1870, 116]. If the dots are very close together, they would appear to be continuous to the general observer and “this is the sort of representation of what we might have to suppose if we did not assume the truth of the law of continuity,” Clifford stated [Clifford 1870, 117]. In that discontinuous world, however, the path from one of the end-dots to the other would be composed of a series of “discrete positions”.

In more technical terminology, Clifford described the mathematical “hypothesis of continuity” as the claim that (see Figure 4),

The motion of N asserts that not merely N itself moves without any jumps, but that the rate at which N is going changes gradually without any jumps, and consequently that the direction of P ’s motion changes gradually; or that the curve described by P cannot have a sharp point. [Clifford 1870, 119]

Thus, one can calculate the rate of change of the curve representing the rate of change of another curve (*i.e.*, a 2nd order derivative). Indeed, the mathematical hypothesis of continuity asserts that this process can be repeated until there is no longer a rate of change to measure. Ultimately, however, these more advanced mathematical aspects of “continuity” are still based upon the physical truth of continuous space:

The hypothesis of the perfect continuity of N ’s motion asserts that all these points move continuously without any jumps. [Whereas] a jump made by any one of these points, being a finite change made in no time, would be a change made at an infinite rate; the next point, therefore, and all after it, would go right away from 0,

FIGURE 4: [Clifford 1870, 118]

and disappear altogether. We may express the law of continuity also in this form; that there is no infinite change of any order. [Clifford 1870, 121]

Ultimately, the hypothesis of continuity states that existent bodies never lose their existence in space.

Thus, for Clifford, mathematical “continuity” is dependent upon what happens to an object in space as time progresses. If space were composed of discrete units, then the movement of the object must involve leaps from one segment of space to another with no travel in-between. It would involve what Clifford would consider to be an instantaneous action-at-a-distance. And if one were to try to measure a “rate of change” based on the size of a jump from one dot to another, one would be led to the consideration that,

This rate might obviously change by jumps as violent and sudden as those of the thing itself; at any instant, when the thing was non-existent, its rate would be non-existent, and whenever the thing came into existence its rate would suddenly have a value depending on how far off its last position was. [Clifford 1870, 118]

In a discontinuous world there could be no such thing as a meaningful “rate of motion.” There would be no meaningful use of the calculus, and certainly no meaningful use of quaternions or vector transformations. As a spatial model, discontinuity would mean that calculus could not accurately represent the world as it is. The measurement of distances between points or bodies in space would become a useless task, as it would tell us nothing about the intervening segments of space. Mathematics as a descriptive science would fall apart, as it would have no role to play in describing the causal picture of the universe. Clifford thus acknowledged that science post-Newton

and post-Leibniz could only move forward on the justifiable *assumption* that continuity holds true in space and time. Otherwise, the entire structure of calculus would have to be done away with and new tools for measuring motion in discontinuous space would have to be developed.¹⁷

7. On Ether as the Necessary Result of the Continuity Hypothesis

Perhaps not surprisingly, Clifford's reliance upon the *necessity* of "continuity" in space, and upon the physiological predisposition of human beings to believe in it, was deeply intertwined with his belief in the existence of the ether—a medium that guaranteed a material world in which cause and effect could be explained via the interactions of particles. From Clifford's perspective, an all-pervading ether precluded the need for metaphysical explanations of physical phenomena. Delivered to the Sunday Lecture Society in 1872 (a popular forum), Clifford's essay, entitled "Atoms", expounded upon the link between "continuity" and the ether and also the link between mystical beliefs in "God" and action-at-a-distance "forces."

Taking the example of a person rubbing a wet finger along the rim of a glass, Clifford argued that in order to explain the phenomena of "sound" (*i.e.*, the perception of musical waves) scientists had to acknowledge the existence of a "surrounding" framework through which waves could travel such that a person distant from the origin of the sound could perceive it. At its most basic level, that surrounding framework (the ether) is composed of "atoms", which pervade the entire universe such that they are always in contact with one another. According to Clifford, this understanding of the universe's continuous physical structure was no longer deniable:

The 'atomic theory' (...) is no longer in the position of a theory, but that such of the facts as I have just explained to you are really things which are definitely known and which are no longer suppositions; that the arguments by which scientific men have been led to adopt these views are such as, to anybody who fairly considers them, justify that person in believing that the statements are true. [Clifford 1872, 163]

The "undulatory theory" of light was a scientific theory that Clifford also considered to be consistent with his view of the universe, as it allowed for the transference of "energy" between contiguous entities with no discontinuous

17. Clifford wrote often about the "fluxion" method that Newton had developed, although it will be clear to any observer that Clifford—as with many mathematicians of the late-19th century—interpreted the Newtonian fluxion and the Leibnizian derivative as two sides of the same coin. Clifford's goal was to link the assumption of "continuity" to the scientific usefulness of the Newtonian fluxional method.

jumps in space. In other words, light consists of waves transmitted through a physically continuous *medium*. Clifford's aim in drawing out this example was to convince mathematicians that if they were to rely upon the fundamentally *physical* notion of continuity, then they would be well advised to also recognize the useful corollary outcomes of that choice. One such outcome was the fact that, in order for light waves to travel from one location in the universe to another, there had to exist a mechanical series of entities between the two positions such that light waves never disappeared and reappeared in the process of transmission. From a Cliffordian perspective, therefore, if scientists and mathematicians hoped to make meaningful claims about the universe, they were compelled to adopt a thorough belief in the continuity of space and time. In so doing, they had to resist conventionalising mathematical knowledge such that "continuity" became a mere logical tool, or symbolical axiom, defined intrinsically and thus severed from the very empirical inferences that had given rise to the notion in the first place.

Conclusion

Clifford's final statement on continuity comes in the form of his monograph, *The Common Sense of the Exact Sciences* (published posthumously in 1885 and edited by Karl Pearson). In his *Common Sense* text Clifford brought together the various strands of his continuity theorizing, linking them to his views on empiricism, physiology, non-Euclidean geometry, and bi-quaternions. In a chapter entitled "Of Mass and Force", he wrote that:

The custom of basing our ideas of motion on these terms "matter" and "force" has too often led to obscurity, not only in mathematical, but in philosophical reasoning. We do not know *why* the presence of one body tends to change the velocity of another; to say that it arises from the force resident in the first body acting upon the matter of the moving body is only to slur over our ignorance. All that we do know is that the presence of one body may tend to change the velocity of another, and that, if it does, the change can be ascertained from experiment.¹⁸ [Clifford 1885, 243]

For Clifford, the question of whether mathematical "continuity" ought to be defined independently of physical continuity is a question about what constitutes good scientific and mathematical theorizing. Given that empirical sensory observation can only tell us so much about the world "in itself", scientists are heavily reliant upon their assumptions about *how it ought to be*.

18. Although Pearson was responsible for heavily editing this section of the book, Clifford makes similar claims in various lectures delivered throughout the 1870s, many of which were later published in his *Lectures and Essays*. Thus, the statement is a fair representation of Clifford's own views on the subject.

For Clifford, therefore, it is sensible to assume continuity exists in the physical structure of the universe and to use this principle in order to define and develop new geometrical models that account for what often appear to be discontinuous events (*i.e.*, gravity and other action-at-a-distance phenomena).

The Cliffordian case study demonstrates that “continuity” has served various scientific, mathematical, philosophical and even physiological roles throughout history. In Clifford’s case, it formed an intricate part of his philosophy of science, and it underpinned his deeply materialist philosophy of space. In comparing his account to that of Dedekind’s arithmetization of the concept, it is possible to highlight the fact that in these divergent interpretations of a fundamental concept in mathematics, a finitist picture emerges—one in which the concept of “continuity” does not come pre-packaged with a series of understood and consensual meanings bound together. Rather, “continuity”, as with all mathematical concepts, is underdetermined. Practitioners are free to idiosyncratically draw upon the wide-spread cultural resources available to them in adopting, redefining and using such concepts in previously unknown ways. In this conceptually underdetermined world, it becomes clear that, for someone like Dedekind, the motivation to define the concept of continuity stemmed from a belief in the abstract and conventional (if not language-like) nature of mathematics as distinct from any physical concerns. For Clifford, on the other hand, the definition and use of continuity stemmed from his thoroughly physics-minded agenda, which ultimately placed limits on the bounds of what could constitute good scientific theorizing (*i.e.*, appeals to materialist descriptions of cause and effect) and good mathematical practice (*i.e.*, the development of descriptive geometrical tools that would preserve continuity and account for physical phenomena). Each practitioner viewed his own account as the correct one; each practitioner thus managed to redefine the bounds of proper use for this seemingly simple and intuitively clear notion.

References

- BARNES, B., BLOOR, D. & HENRY, J.
 1996 *Scientific Knowledge: A Sociological Analysis*, Chicago: The University of Chicago Press.
- BENARDETE, J. A.
 1968 Continuity and the Theory of Measurement, *The Journal of Philosophy*, 65(14), 411–430.

BOI, L.

- 1997 Géométrie elliptique non-euclidienne et théorie des biquaternions chez Clifford : l'élaboration d'une algèbre géométrique. *Le Nombre, une hydre à n visages : Entre nombres complexes et vecteurs*, Paris : Fondation Maison des Sciences de l'Homme, 209–238.

CANTOR, G. N. & HODGE, M. J. S.

- 1981 *Conceptions of Ether: Studies in the History of Ether Theories 1740-1900*, Cambridge: The Cambridge University Press.

CLIFFORD, W. K.

- 1870 On Theories of Physical Forces, *Lectures and Essays*, London: MacMillan and Co., 1879, 109–123.
- 1872 Atoms, *Lectures and Essays*, London: MacMillan and Co., 1879, 158–189.
- 1873 Preliminary Sketches of Bi-quaternions, *Mathematical Papers*, London: MacMillan and Co., 1882, 181–196.
- 1876a Further Note on Bi-quaternions, *Mathematical Papers*, London: MacMillan and Co., 1882, 385–394.
- 1876b On the Space-Theory of Matter (Abstract), *Mathematical Papers*, London: Macmillan and Co., 1882, 21–22.
- 1878 *Elements of Dynamic, An Introduction to the Study of Motion and Rest in Solid and Fluid Bodies, Part I. Kinematic*, London: Macmillan and Co.
- 1879 The Philosophy of the Pure Sciences, *Lectures and Essays*, London: MacMillan and Co., 301–409.
- 1882 *Mathematical Papers*, London: MacMillan and Co.
- 1885 *Common Sense of the Exact Sciences*, London: Sigma Books, 1947.

COOKE, R.

- 2005 Richard Dedekind, *Stetigkeit und Irrationale Zahlen* (1872), *Landmarks Writings in Western Mathematics, 1640-1940*, Amsterdam: Elsevier, B.V., Chapter 43, 553–563.

CROWE, M. J.

- 1967 *A History of Vector Analysis: The Evolution of the Idea of a Vectorial System*, Notre Dame and London: University of Notre Dame Press.

DEDEKIND, R.

- 1963 Continuity and Irrational Numbers. *Essays on the Theory of Numbers*, New York: Dover, 1–27.

EVANS, M. G.

- 1955 Aristotle, Newton and the Theory of Continuous Magnitudes, *Journal of the History of Ideas*, 16(4), 548–557.

FOWLER, D.

- 1992 Dedekind's Theorem: $\sqrt{2} \times \sqrt{3} = \sqrt{6}$, *The American Mathematical Monthly*, 99(8), 725–733.

GRAY, J.

- 2005 Bernhard Riemann, Posthumous Thesis 'On the hypotheses which lie at the foundation of geometry' (1867). *Landmark Writings in Western Mathematics, 1640-1940*, Amsterdam: Elsevier B.V., Chapter 39, 506–520.

JOURDAIN, P. E. B.

- 1908 The Introduction of Irrational Numbers, *The Mathematical Gazette*, 4(69), 201–209.
- 1867 Law and Bye Laws of the Cambridge Philosophical Society, *The Cambridge University Calendar For the Year 1866*, Cambridge: Deighton, Bell, and Co., 591–605.

LEWIS, A. C.

- 2005 William Rowan Hamilton, Lectures on Quaternions (1853). *Landmark Writings in Western Mathematics, 1640-1940*, Amsterdam: Elsevier B.V., Chapter 35, 460–469.

LIGHTMAN, B.

- 1997 "The Voices of Nature": Popularizing Victorian Science, *Victorian Science in Context*, Chicago: University of Chicago Press, Chapter 9, 187–211.

MEULLER, I.

- 1969 Zeno's Paradoxes and Continuity, *Mind, New Series*, 78(309), 129–131.

PITKIN, W. B.

- 1906 Continuity and Number, *The Philosophical Review*, 15(6), 597–605.

PYCIOR, H. M.

- 1976 *Ph.D. Thesis, The Role of Sir William Rowan Hamilton in the Development of British Modern Algebra*, Cornell University.
- 1997 *Symbols, Impossible Numbers, and Geometric Entanglements: British Algebra Through the Commentaries on Newton's Universal Arithmetick*, Cambridge: Cambridge University Press.

RIEMANN, B.

- 1868 Ueber die Hypothesen, welche der Geometrie zu Grunde liegen, *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, 13, 1867. Cited according to the English translation by William Kingdon Clifford, *On the Hypotheses which Lie at the Bases of Geometry*, *Mathematical Papers*, London: MacMillan and Co., 1882, 55–71.

SALANSKIS, J.-M. & SINACEUR, H.

- 1992 *Le Labyrinthe du Continu : Colloque de Cerisy*, Paris : Springer-Verlag.

STEWART, J.

- 2003 2.5 Continuity, *Calculus, 5th Edition*, Belmont: Brooks/Cole - Thomson Learning, 102–110.

TUCKER, R.

- 1882 Preface, *Mathematical Papers*, London: MacMillan and Co., xv–xxix.

VOORHEES, B. & CHUN, C. S.

- 2004 *Introduction to Calculus 1: Mathematics 265 Study Guide*, Lethbridge: Athabasca University.

WARWICK, A.

- 2003 *Masters of Theory: Cambridge and the Rise of Mathematical Physics*, Chicago: University of Chicago Press.

WHITE, HENRY S.

- 1916 Mathematics in the Nineteenth Century, *Science, New Series*, 43 (1113), 583–592.

WHITROW, G. J.

- 1933 Continuity and Irrational Number, *The Mathematical Gazette*, 17 (224), 151–157.