Philosophia Scientiæ

Travaux d'histoire et de philosophie des sciences

25-2 | 2021

Mathématique et philosophie leibniziennes à la
lumière des manuscrits inédits

Leibniz's Binary Algebra and its Role in the Expression and Classification of Numbers

Mattia Brancato

Electronic version

URL: <https://journals.openedition.org/philosophiascientiae/3045>

DOI: 10.4000/philosophiascientiae.3045

ISSN: 1775-4283

Publisher

Éditions Kimé

Printed version

Date of publication: 25 May 2021

Number of pages: 71-94

ISBN: 978-2-38072-015-0

ISSN: 1281-2463

Electronic reference

Mattia Brancato, "Leibniz's Binary Algebra and its Role in the Expression and Classification of Numbers", *Philosophia Scientiæ* [Online], 25-2 | 2021, Online since 12 July 2021, connection on 04 November 2021. URL: <http://journals.openedition.org/philosophiascientiae/3045> ; DOI: <https://doi.org/10.4000/philosophiascientiae.3045>

Tous droits réservés

Leibniz's Binary Algebra and its Role in the Expression and Classification of Numbers

Mattia Brancato

Centre Gilles-Gaston-Granger, UMR 7304 Aix-Marseille
Université, Aix-en-Provence (France)

Résumé : Le système numérique binaire élaboré par Leibniz est généralement étudié pour son intérêt arithmétique. L'analyse de plusieurs manuscrits inédits montre, cependant, que Leibniz envisageait aussi dans le contexte de sa dyadique une nouvelle forme d'algèbre, fondée sur l'idée que les lettres pourraient n'exprimer que des nombres qui seraient 1 ou 0. Dans cet article, je présente les deux principaux résultats de cette algèbre binaire : la détermination de l'algorithme pour le développement des carrés et l'élaboration d'une notation positionnelle pour l'expression de tous les nombres possibles. Ces accomplissements expliquent la prévalence, dans l'esprit de Leibniz, de la dyadique sur tous les autres systèmes numériques et son impressionnant travail sur la classification des nombres, favorisé par des outils et des notions dont lui seul disposait à cette époque.

Abstract: Leibniz's binary numeral system is generally studied for its arithmetical relevance, but the analysis of several unpublished manuscripts shows that from the very beginning Leibniz also envisaged a new form of algebra in the context of dyadics based on the idea that its letters can only express numbers that are either 1 or 0. In this paper, I shall present the most notable results of this binary algebra: the determination of the algorithm for the expansion of squares and the development of the positional notation used to express any possible number. These achievements explain the priority of dyadics over any other numeral system in Leibniz's mind and his impressive work on the classification of numbers which was facilitated by tools and notions which were only accessible to him at that time.

1 Introduction

Leibniz's *De progressionē dyadica* [LHXXXV, 3, B 2, fol. 1–4] is probably the most famous, yet the most overlooked unpublished manuscript concerning the development of the binary numeral system. It is known by anyone who is even remotely interested in the evolution of Leibniz's dyadics, because it is dated by Leibniz himself (15 March 1679), thus considered the earliest reliable document about this topic, and at the same time, no one has already offered a complete transcription of it,¹ probably because, at a glance, it doesn't present anything particularly relevant or out of place: as is expected from a work conceived to introduce a new numeral system, it starts by presenting the new base-2 way of counting, confronting it with the traditional base-10 system, finding a way to express any base-10 number as its correspondent base-2 number, and then concluding by introducing the four basic arithmetic operations of addition, subtraction, multiplication and division in their binary fashion.

It is, however, less known that this manuscript is actually composed of two parts, where the first relatively straightforward part of it, that confirms our expectations concerning dyadics, is followed by an obscure second part in which Leibniz shifts his attention from arithmetic to algebra. This last part would appear puzzling to scholars interested in the development of the binary system, because algebra is primarily the study of mathematical symbols and quantities in their most general form, while the novelty of Leibniz's dyadics should lie in its arithmetic. As a matter of fact, if the goal of algebra is introducing abstractions, such as letters that stand for numbers that are unknown or that could represent any value, to highlight the relationship between these entities in spite of their actual value, a binary number expressed in such form should ideally lose all the specific properties that make the binary form of expression unique. Moreover, the numerical part of the algebra presented by Leibniz in the manuscript is written in its decimal form, while only the generic letters represent binary quantities in his mind, resulting in multiple hybridisations difficult to understand: decimal coefficients operate on letters that are not allowed to express any quantity in any fashion, but only the binary numbers 1 and 0. This peculiar kind of "binary algebra"² has never been thoroughly analysed by Leibniz scholars, due mainly to the unavailability of primary sources.

1. The manuscript is in fact already mentioned by Couturat [Leibniz 1903, 574]. In 1966, a facsimile of the manuscript and an incomplete German translation appeared in [Leibniz 1966], mentioning only its first part, with no transcription or translation of its second part.

2. Even if this term is not explicitly used by Leibniz, he is very aware of the uniqueness of his approach in comparison to standard algebra, as it will be clear for instance from the quote at the end of this paper. I propose the neologism "binary algebra" to effectively differentiate it both from binary arithmetic and standard algebra.

In this paper, by explaining the second part of *De progressionē dyadica* through the analysis of other relevant manuscripts, I will argue that understanding Leibniz's binary algebra allows us to understand also the mathematical importance of the binary system in Leibniz's mind, the reason why he developed it and the connection between its mathematical relevance and its philosophical relevance. This study will connect dyadics with Leibniz's reflections on the nature of numbers, whether they are natural, rational or irrational, and the possibility of expressing them in a way that allows a better *a priori* representation through the use of a positional and combinatorial method.

It has already been shown³ that the birth of the binary numeral system was connected from the very beginning with its metaphysical implications, even if *De progressionē dyadica* doesn't mention any specific metaphysical topic. In the wake of a renewed interest in classic Platonic and Pythagorean themes perpetrated by Weigel and his scholars around that time, Leibniz, as a former student under Weigel, saw in the binary numeral system the perfect expression of a metaphysical truth, that of the composition of reality through the combination of different degrees of being and nothingness, represented respectively by the numbers 1 and 0.⁴ For the purpose of maintaining the connection with the mathematical development, Leibniz conceived the creation of the world as a computational procedure made by God's infinite mind, the only one capable of processing the infinite amount of data composed of things expressed in their dyadic form, where the number of ones and zeros belonging to every existence indicate the unique level of being and nothingness needed to shape a particular entity in that specific way. This is the main assumption behind some of Leibniz's claims connected to dyadics, such as the one contained in another fundamental manuscript, the *Summum calculi analytici fastigium per calculum algorithmicum* [LH XXXV, 13, 3, fol. 21–22], dated December 1679, few months after *De progressionē dyadica*:

we assume here the binary progression, that is the dyadic one that the nature of things itself prescribed: all the others are indeed arbitrary. [Zacher 1973, 218]⁵

While Leibniz's predilection for the binary system is amply justified by the metaphysical reasons highlighted throughout his whole life, it would be equally

3. The metaphysical background was influenced mainly by Weigel's book *De supputatione multitudinis a nullitate per unitates finitas in infinitum collineantis* [Brancato 2016].

4. The topic of expression at a metaphysical level is of course much wider than dyadics in Leibniz's philosophy, but the problem of the compatibility between the general concept of expression and the one assumed in dyadics still needs to be addressed by scholars. On the general notion of expression see [Debuiche 2019].

5. As in this case and unless otherwise indicated, the translations of passages published only in Latin and the transcription and translation of unpublished manuscripts were made by the author of this paper.

important to understand if there are specific mathematical reasons for this preference,⁶ even more if at a philosophical level the distinction between God's mind and that of finite creatures introduces a degree of unverifiability that is unlikely acceptable in the light of Leibniz's rationalism.

If understood correctly, Leibniz's binary algebra offers exactly this kind of mathematical grounding. It is through binary algebra in fact that the binary system achieves one of the greatest mathematical results in Leibniz's eyes: the possibility of determining something in general about the succession of digits of any possible number, including irrational numbers. The explanation of this fundamental result is contained, in one of its most mature forms, in the *Summum calculi analytici fastigium*, making this manuscript the ideal starting point to understand the relevance of *De progressionem dyadica*'s rarely analysed second part. Before presenting in detail Leibniz's reasoning however, it is important to show how all the elements involved in his analysis were already part of a wider and cohesive mathematical program, developed before 1679.

2 The premise to the *Summum calculi analytici fastigium*'s classification of numbers

The *Summum calculi analytici fastigium* was already transcribed by Hans Zacher in his seminal work on dyadics in 1973, but the complicated diffusion of this book has prevented its serious consideration by many scholars. In this manuscript, Leibniz envisions the possibility of expressing every type of number through the use of binary series. This attempt presupposes that even for infinite yet non-periodic series, like the ones generated by irrational numbers, we can deduce in some way the succession of their digits. In trying to set the general rules for this deduction, Leibniz shows an interesting and peculiar use of binary algebra as a tool, in conjunction with other achievements of his mathematics.

From the beginning of the manuscript, however, it is already clear that these kinds of explorations on the nature of numbers presuppose a very informed awareness about their properties:

Every fractional, irrational and transcendent quantity can be expressed through a decimal number prolonged to infinity. [Zacher 1973, 218]

6. Leibniz has surely thought about dyadics as something interesting in and of itself. Yet, thinking about this topic as something completely separated from the mathematical problems that he was facing at the time of its development is unlikely: studying dyadics as a tool is then the key to understanding its birth.

Rational numbers, Leibniz continues, can produce periodic series, while for irrational ones the series won't be periodic, even if it can still be investigated in a certain way. Such a precise distinction between different kinds of numbers derives from the summation of Leibniz's mathematical efforts before 1679: from the studies on number series, polynomials and Descartes' analytic geometry to the ones on Wallis' works and on imaginary numbers, carried out together with Tschirnhaus. Above all and setting aside Weigel, the confrontation with Wallis seems to be the most relevant in this regard: his works are, in fact, mentioned by Leibniz in two important and almost coeval dialogues that appeared few months before *De progressionē dyadica*, the *Dialogi familiares de arcana mathematicorum analysi* [LH XXXV, 8, 30, fol. 35], a fragment dated 1678, and the *Dialogue on arithmetic and algebra* [LH XXXV, 1, 29, fol. 1–19], a manuscript dated around the end of 1678 and already transcribed by Eberhard Knobloch [Leibniz 1976]. These dialogues contain many of the elements present in Leibniz's early writings on dyadics, except the core ideas of dyadics itself, making them valuable documents that show how the binary system was most likely conceived in this context, but only after the end of 1678: it wouldn't have made sense for Leibniz, in fact, to compose works on such topics without mentioning the same binary system which, as it will be shown, helped him so much in clarifying some aspects of the distinctions between the different types of numbers presupposed in these dialogues.

The *Dialogi familiares* show how Leibniz was already reflecting on what it means to express a number, even if the idea of changing the number base was not present in his mind yet. Nonetheless, we find here a definition of value that could be interpreted as an ideal premise to Leibniz's further developments:

Value is the expression of a quantity through another equal quantity, expressed with a formula of other quantities. [LH XXXV, 8, 30, fol. 35]

It is important to notice that this definition is applied from quantities to other quantities, and not to unknown variables, as in a basic example like $7 = 3 + 4$, or in the expression of a fraction in its decimal value. This simple principle will be particularly important in Leibniz's reflection on the expression of any possible number because the determination of their values will be grounded in the possibility of expressing them as a sum of finite or infinite series. In this context, Leibniz mentions Wallis' interpolations, a method used to compute quantities that appeared in the *Arithmetica infinitorum*, meaning that at this time he was already reflecting on how to express the value of the irrational number π through the use of infinite series,⁷ a topic already well studied by scholars, that was possibly influenced also by the reception of Gregory.⁸ While being only a small fragment, this dialogue shows Leibniz's interest in

7. These explorations are well documented in [Probst 2015, 129–130].

8. A more detailed analysis of this topic will be presented in the last chapter of this paper. An explanation of Wallis' computation of π is present in [Berggren, Borwein

presenting a great work on mathematics modelled after Wallis' works, which would have started from the concept of value, only to introduce then the four basic operations, algebra and the topic of number series.

This approach resonates with the *Dialogue on arithmetic and algebra*, which can be seen as a partial realisation of the great work only sketched in the *Dialogi familiares*. The connection with Wallis is here even more evident and it might be directed towards a specific work, the *Mathesis universalis*.⁹ Apart from being indirectly mentioned in the *Dialogue*,¹⁰ from Wallis' *Mathesis universalis* Leibniz derives the very structure that has to be adopted in introducing mathematics, not so much because predictably arithmetic is followed by the introduction of algebra, but because everything is preceded by a section called by Leibniz "numeration", or "enumeration", that concerns the art of writing and enunciating numbers, as a first step into arithmetic [Leibniz 1976, 17]. By addressing the problem of counting and decomposing decimals, this first part clearly mimics an extensive part of Wallis' *Mathesis universalis*, where the author thoroughly analyses the base-10 system and its properties, also from a historical point of view, showing examples of the notation and naming in the Roman, Hebrew and Arabic cultures [see Wallis 1657, 21–60]. Since Wallis also suggested the idea of changing the base of counting [see Wallis 1657, chap. X], it is evident that these explorations, especially for their relativistic and historical scope, must have been an important premise to Leibniz's development of the binary numeral system.

Apart from this influence specifically tied to dyadics, in Wallis' work Leibniz could have found also the definitions of natural, fractional, rational and irrational numbers presupposed at the beginning of his *Summum calculi analytici fastigium*. However, the fact that the *Dialogue on arithmetic and algebra* is to a certain extent modelled after the *Mathesis universalis* doesn't mean that Leibniz also follows theoretically Wallis' theories on the classification of numbers. Even if the strictly mathematical characterisation of numbers is the same, i.e., irrational numbers are still numbers that cannot be constructed from ratios of integers, Leibniz argues against some of Wallis' views, to offer a more ambitious and general take on the notion of number.

et al. 2004, 68]. See also [Osler 2010]. James Gregory's work was already known by Leibniz since 1673 through Huygens, see for example [Serfati 2018, 11], but his actual influence on this specific topic is still debated [see Crippa 2019, 100–102] and [Probst 2015, 131].

9. The topic of Leibniz's confrontation with Wallis is of course much wider and it is present already in the Parisian period. On Wallis' *Mathesis universalis* and on the concept of *mathesis universalis* in general, see David Rabouin's introduction in [Leibniz 2018, 29]. On Leibniz's general reception of Wallis, see [Probst 2018].

10. While there exists a copy of the 1657 edition of Wallis' *Mathesis universalis* in the Gottfried Wilhelm Leibniz Bibliothek, that copy entered in the collection through the Huygens inheritance, making it impossible to determine if Leibniz had access to Wallis' work. However, some references in the *Dialogue* suggest so: the analytical explanation of some theorems belonging to *Elements'* book II and the explanation of powers closely resemble that of Wallis' *Mathesis*.

In order to maintain the classic definition of number as a collection of unities, Wallis accepted the idea that there is a fundamental distinction between different kinds of numbers: in his terms, fractional numbers, much like irrational numbers, are not real numbers. Furthermore, to remove any other possible difficulty, decimal numbers that present an infinite periodicity are treated as if they were approximated to the closest non-periodic value.¹¹

Against this traditional approach, Leibniz is happy to dismiss the definition of number as a collection of unities, if a more inclusive definition is possible. The new definition is connected with one of the most interesting passages of the *Dialogue on arithmetic and algebra*, where Leibniz argues that there is a difference between an equation and a ratio: the ratio between two quantities b and c simply requires c to be expressed in a way in which b will be its measure or unity, so that “the ratio is something homogeneous to a number, something which can be added or subtracted to it” [Leibniz 1976, 115]. This passage resonates with what will be Leibniz's definition of number for his entire life: number is something which is homogeneous to unity. The fact that in the definition of number, i.e., something that concerns its nature, a term like “homogeneity”, i.e., something that concerns instead its expression, has a predominant role, ties together these two topics in an unprecedented way, not so much for the definition in itself, which was not completely new at that time, but for the way in which Leibniz rigorously adheres to it. If numbers are not collections of unities as Wallis believed, it also means that natural numbers have to be homogeneous to unity in a similar way in which other numbers are. While it is true that there is an important difference between a reflection on the nature of numbers and a reflection on the possibility of expressing them, so much so that Leibniz himself, just like Wallis, establishes a difference between natural numbers, called numbers *stricto sensu*, and other numbers *lato sensu* in the *Nouveaux Essais*, I believe that Leibniz's most experimental reflections, especially in the early 1670s, come from the opposite idea that the homogeneity in the expression tells us also something about a possible common nature. This is especially true in the context of dyadics, where the expression of numbers has a metaphysical grounding, because it is not any possible expression, but the expression God itself chose, i.e., the real expression of numbers. This tension between the use of fictional numbers and a metaphysically grounded homogeneity might be useful to understand the role of dyadics, which was always enthusiastically pursued by Leibniz, but at the same time hardly implemented in a coherent way with his later mathematical efforts on other topics.

For these reasons then, Wallis' attempt to establish a distinction between real and fictional numbers, mirroring the distinction between whole and fractional numbers or between rational and irrational numbers, has to be reframed in the experimental context which led to dyadics, where instead

11. On the definition of number see [Wallis 1657, 15], on irrational, fractional and decimal numbers see [Wallis 1657, 252, 363, 390].

the common properties among fractional, rational and irrational numbers are perceived as signs of their common nature, as Leibniz explored from 1679.

In the *Dialogue on arithmetic and algebra*, we also find a premise to the binary algebra introduced in *De progressionem dyadica*, or at least a premise to its possibility: “countless *calculi* can be devised, clearly different from the algebraic one and having their own uses” [Leibniz 1976, 57]. Since Leibniz sees algebra as one specification of a wider logical and combinatorial calculus, conceiving a modified algebra, where the letters can only indicate either the number 1 or the number 0 was a possibility soon to be explored.

The reflection on the notion of value and its expression, the homogeneity in the definition of number and the studies on number series all contributed to Leibniz’s attempt of expressing any possible number through his binary algebra in the *Summum calculi analytici fastigium*, once the binary numeral system was introduced. It could be argued then that its development is a direct response to the reflection on these topics, a coherent hypothesis that frees dyadics from the impression of being a side project, independent from Leibniz’s other mathematical explorations of that time.

3 Binary algebra and Leibniz’s positional notation

As was shown, at the beginning of the *Summum calculi analytici fastigium* Leibniz states that any possible number can be expressed as a series, but this terminology and the aim behind its use is unclear, because he immediately shifts his focus from the idea of a series of digits to the idea of a number series, in the sense of a series of sums. While it is true that a number can be expressed as the sum of other numbers, the connection with the succession of digits can be understood only if we intend the series of sums not as any possible sum that results in the value considered, but as a specific sum having a fixed order. For example, the number 324 can be expressed as the sum $300 + 20 + 4$, or the number 1.502 as the sum $1 + 0.5 + 0.00 + 0.002$. In other words, Leibniz is using in this context what nowadays is called a positional/place-value notation or, as called in some mathematical traditions, the polynomial expression of a number. In this way, every number can be expressed showing the base of the numeral system adopted, where the powers of the base indicate the position of the digit in the succession:

$$324 = (3 \times 100) + (2 \times 10) + (4 \times 1) = (3 \times 10^2) + (2 \times 10^1) + (4 \times 10^0)$$

The *Summum calculi analytici fastigium*, and before it *De progressionem dyadica*’s second part, is a study on the properties of a positional notation, when the base considered is 2, i.e., when the number is expressed in the

binary numeral system.¹² However, this approach is not clear in the text and it caused some misunderstandings among the interpreters that prevented the appreciation of its mathematical relevance. The reason is that Leibniz does not give any arithmetical example of his notation, starting immediately with an algebraization that doesn't show clearly the number base used, making the confusion with an ordinary sum very easy. Instead, Zacher [see Zacher 1973, 60–64] recognised the use of the positional notation, but he believed that Leibniz focused only on the number base, thus making a mistake that prevented him from achieving any actual progress on this topic from the *Summum calculi analytici fastigium* onwards. An analysis of Leibniz's method will show instead that, despite some minor mistakes¹³ and a notation not always up to the task, a relevant mathematical result was achieved from the very beginning.

Leibniz tries first a process of algebraization by substituting letters to numbers as in the example $\dots e + d + c + b = a$, where the letter a stands in this case for the ordinary expression of the designated quantity, while the other letters stand for its digits expressed in the positional notation as a sum, considering the base and its powers. Leibniz inverts the equivalence to highlight in a better way the fact that the result can be ideally prolonged to infinity: he is able in this fashion to show the development of the succession as much as he wants without being hindered by the symbol indicating the prosecution to infinity, which is now relegated to the external side of the manuscript. With this method, we could ideally represent any possible succession, even periodic successions, but Leibniz's aim here is clearly the study of the natural numbers from 0 to infinity: a shift from b to c would mean a shift of the base from 2^0 to 2^1 , and so on, the higher the number is. Leibniz's claims that he is able to express in this notation also fractional and periodic rational numbers are nonetheless completely valid, provided we keep in mind that the part after the radix point will express bases with a negative exponent and that the properties Leibniz will find for numbers before the radix point are also valid for the numbers after the radix point, only following a pattern which is symmetrical to the first one. If our aim then would be showing the behaviour of the progression of a rational number having a period that proceeds to infinity, we would just need to adopt a notation symmetrical to that of Leibniz, so that the prosecution to infinity will describe the behaviour of the part after the radix point.

In any case, Leibniz realises that this notation is not sufficient for his purposes, because some nuances of his binary algebra are here lost. As was

12. The idea of changing a base to express something more easily was probably connected to Leibniz's studies on logarithms, since at that time he was already aware of the works of Napier, Briggs, and, above all, Kepler.

13. For example, Leibniz shows a fluctuation in the choice of the variables or a wrong use of his own new notation, e.g., the use of the number 20 instead of the number 110 as the successor of the number 19, not following the notation that he himself introduced and that will be explained below.

briefly mentioned in the introduction of this paper, the main premise behind a form of binary algebra is not only that every letter must represent a binary number, but also that every letter can only represent the number 1 or the number 0. This premise would be true only if the number is expressed as a series of subsequent digits, but it doesn't hold if the same series is expressed as an ordinary sum. In the example of the number 324, its binary expression would be 101000100 and a possible algebraization of it would be *lihgfedcb* = 101000100. In this case, every letter, provided we could conceive infinitely many of them for larger numbers, would effectively indicate a number which is either 1 or 0, so that they would represent digits and not numbers because otherwise, that succession of letters would indicate a product. Much like it was shown for a decimal sum, however, the same doesn't apply for the correspondent binary sum, because in this case, for example, the letter *b* would be equal to 0, *c* to 00, *d* to 100, *e* to 0000, *h* to 1000000, and so on: binary numbers indeed, but not restricted to 1 or 0.

In order to maintain this restriction and to maintain also the general series in the form of a sum, Leibniz invents a new notation based on the idea that we can still consider the symbols used only as single ones or zeros, if the notation adopted mentions also the position of the digit in the succession. To achieve this goal, Leibniz utilises a notation based on decimal numbers, so that the general series expressing the number will be in the form:

$$\dots 111 + 110 + 19 + 18 + 17 + 16 + 15 + 14 + 13 + 12 + 11 = b$$

In this new notation, the first number used (1 in this case) is always the same and it indicates to which series the digit belongs, so that we can easily recognise it if a comparison with another series is needed. If we were to introduce a series of another number then, it would be written in the form $\dots 23 + 22 + 21 = c$.

The second number of the new notation instead represents the position of the digit in the sum. It is in this sense that the number following 19 is 110 and not 20, since the first number acts as separated from the indication of the order. In the example above then, the symbol 13 does not represent anymore, as was the case for the letter *d*, the number 100, but the number 1, provided that we maintain the order granted by the new notation and put that number at the third place in the succession. In this context then, the idea of order becomes extremely important, because even if we are considering a sum, the commutative property does not apply as usual: if the number 1, for instance, appears at the position 15, a shift from the position 15 to the position 16 would mean that we are not dealing anymore with the number 10000, but with the number 100000 or, to follow strictly the new notation, the number 1 at the position 5 in one case and the number 1 at the position 6 in the other. This way of associating numbers having different functions was already explored by Leibniz in other writings, where they are defined as fictitious numbers [see for instance Knobloch 2010, 296]. At the price of turning the expression of

the series in a sum having a fixed order, where every shift has to be justified, Leibniz successfully finds a way to maintain its form as a sum and the rule by which every symbol expresses either the number 1 or the number 0. In contemporary terms, the second number of the new notation expresses the exponent of the base, provided we keep in mind that the symbol 11 represents the base 2^0 (also called the index 0), so that the numbers used by Leibniz are all shifted to one place with respect to the positional notation we would use.

It is important to remark that Leibniz is still moving here in the realm of algebra: even if the new notation is composed by numbers, they are utilised as symbols, because they inform on certain relationships among the series, but they are not in any case mutually computable, as if they were common decimal numbers: writing $12 + 11$ is the same as writing $a + b$, but it wouldn't make sense to state that the result of the sum is 23, or even 13.

Leibniz's efforts are all guided by the idea that if we are able to write any general series in this new notation, then we can compare two series that have some connection with each other and state something universally valid about them. In particular, Leibniz compares any general series with its square. Since he successfully maintained the expression through a sum, the basic notions on the properties of the square of $(b + a)$ in the form $(b^2 + 2ab + a^2)$ can still be applied. Being however a possibly infinite sum that has a fixed order, the case considered by Leibniz forces him to establish also in which position all the elements of the developed square reside. He does so by writing a first line that starts with the square of the first element (counting from the first digit, i.e., from right to left) and continues indefinitely with all the possible double-products between the first element of the series and all the other members of it. Then, he repeats the process for the second element in the series, only shifted by two places, and proceeds indefinitely until the sum of all the lines gives the expression of the squared series, using however the terms belonging to the original series. If we express any series indicating a number in the form $\dots 15 + 14 + 13 + 12 + 11 = b$ and the series indicating its square as $\dots 28 + 27 + 26 + 25 + 24 + 23 + 22 + 21 = c = b^2$, considering the following commas as the multiplication operator in Leibniz's notation and the circled numbers as the coefficients, the expression of the second series through the use of the numbers contained in the first series will be calculated as follows:¹⁴

14. The following table appears in the *Summum calculi analytici fastigium* [Zacher 1973, 220]. It has been modified in several aspects for a better understanding of Leibniz's goals: the so-called balance beam symbol used by Leibniz for equality has been substituted with the common one and the entry "11,11" in the first column, missing in the original manuscript [LHXXXV, 13, 3, fol. 21v], has been also added. Moreover, for space reasons, two columns on the left which are present in the manuscript are here missing.

IX	VIII	VII	VI	V	IV	III	II	I	
15^2		14^2		13^2		12^2		11^2	
$\textcircled{2}11,19$	$\textcircled{2}11,18$	$\textcircled{2}11,17$	$\textcircled{2}11,16$	$\textcircled{2}11,15$	$\textcircled{2}11,14$	$\textcircled{2}11,13$	$\textcircled{2}11,12$	$11,11$	$=b^2(\text{A})$
$\textcircled{2}12,18$	$\textcircled{2}12,17$	$\textcircled{2}12,16$	$\textcircled{2}12,15$	$\textcircled{2}12,14$	$\textcircled{2}12,13$	$12,12$			
$\textcircled{2}13,17$	$\textcircled{2}13,16$	$\textcircled{2}13,15$	$\textcircled{2}13,14$	$13,13$					
$\textcircled{2}14,16$	$\textcircled{2}14,15$	$14,14$							
$15,15$									
$11,18$	$11,17$	$11,16$	$11,15$	$11,14$	$11,13$	$11,12$		11	$=b^2(\text{B})$
$12,17$	$12,16$	$12,15$	$12,14$	$12,13$		12			
$13,16$	$13,15$	$13,14$		13					
$14,15$		14							
15									
29	28	27	26	25	24	23	22	21	$=b^2(\text{C})$

Leibniz obtains two algorithms describing the development of the square of any possible number in his positional notation. The first one (A) is valid for any number base adopted, the second one (B) is valid only in a base-2 notation, that is conceiving binary quantities behind the algebra used. The third expression of the square (C) is equal to the simple succession of digits of the square chosen. The table compiled by Leibniz contains a mistake, corrected in the table presented here: the digit at the second column (22) is omitted and the counting of the digits resumes from the third column, so that the symbol 22 is placed instead of 23, 23 instead of 24, and so on. This mistake was not made in *De progressionē dyadica* and it is partially justified here by reasons that will be soon analysed, but it is nonetheless responsible for the misinterpretation of this passage and its consideration as irrelevant.

Already the first algorithm (A), which can be applied for any number base, represents a mathematical achievement: it is the expression in a positional notation of the expansion of any possible square. It conveys more information with respect to the traditional expansion of the square because it informs on which parts of the sum are involved in the determination of a specific digit. Adopting x to express any possible base, the general expansion of two elements b and a could be rewritten as follows: $b^2x^2 + 2abx^1 + a^2x^0$. An ordinary expansion would present ordinary additions, therefore, while the result would be the same, it would be possible to apply the commutative property without caring for the position of the digits, thus we wouldn't know which operations are exactly involved in the determination of every single digit of the square, which is instead what Leibniz achieves in (A).

The mathematical result of the second algorithm (B) is even more notable because it shows an alternative expansion of any possible square, valid only for binary numbers and obtained from the first algorithm by executing all the calculations present in (A) in the case in which they represent specifically binary numbers. The usefulness and superiority of Leibniz's binary algebra is finally evident and it is represented mainly by two results. The first one concerns the square of any number: as it is shown in column I for (B), the product of a number and itself found in any column in (A) is always equal to the number itself in the same column in (B). This is possible only if the numbers considered are binary numbers, because 11×11 (11, 11 in Leibniz's notation) is equal to 1, if 11 represents the number 1, and it is equal to 0 if 11 represents the number 0, so that every time there is a number multiplied by itself, we are allowed to write that number instead of its square. In Leibniz's words:

this binary [progression] is indeed a great turning point because any of those characters is 1 or 0. For this reason, the powers of the assumed letters, for instance a^2 , b^2 , etc., as much as a^3 , b^3 , etc. are the same as the roots, for instance $a^2 = a$, $b^2 = b$. So, in any case, we grow higher in the equations of this calculus, in vain however we will grow, and everything will be dealt with only through pure rational equations. [Zacher 1973, 218]

The second relevant observation on the general table concerns the double-products: Leibniz discovers that every time there is one in the first algorithm (A), we are allowed to write it shifted by one position from right to left in the binary one (B), eliminating the coefficient 2. For example, ②11,12 in column II for (A) is reported as 11,12 in column III for the second algorithm (B). Once again, this is a direct consequence of the binary system that lies beneath this algebra: if one of the symbols between 11 and 12 stands for 0, then their product will be 0, whereas if both terms stand for the number 1, then the product will be 2, i.e., 10 in binary notation, representing a shift from the second column to the third one.¹⁵ Considering all these elements, we can carelessly write 11,12 in the third column of algorithm (B) instead of writing ②11,12 in the second one as it happens in algorithm (A), because in the case in which the product is 0 it would be as writing something uninfluential and in the case in which the product is 1 then it means that an actual shift between the second and the third column happened and that value has to be 10 in the second column, that is 1 in the third. This example shows how Leibniz's notation presented before has not yet been described completely because strictly speaking the number ②11,12 would be equal in today's positional notation to $2(11 \times 12) \times 2^1$, while the same product in the third position would be equal to $(11 \times 12) \times 2^2$: the determination of the numerical value of the square's third digit depends on the product between the first two digits of the root indeed, but only if it is taken with the appropriate exponent of the base. Leibniz's notation then is not composed only by the symbols 11, 12, 13, 14..., but it is a combination between them and the roman numbers at the top of the table, indicating the exponent of the base. Following this kind of algebra is admittedly hard, hence the many misinterpretations.¹⁶

Since Leibniz doesn't offer any example of his results, two specific applications of the algorithms (A) and (B) are presented below, for a base-10 number in the first case and for a base-2 number in the second case, both converted to the positional notation used:

15. This situation reveals Leibniz's awareness concerning the hybrid model of his binary algebra: the coefficients pose no problems in their decimal form, only until they are not involved in the calculations with the binary numbers expressed as symbols, but the moment the actual calculation happens and these symbols are supposed to assume the values 1, 0, or "a value that is 1 or 0", the coefficients start to behave as binary numbers as well. In this case, in fact, the number 2 wouldn't cause a shift in the positioning, if it were to be considered a number expressed in its decimal form.

16. This analysis shows that Zacher's interpretation has to be rejected, because Leibniz's algorithm is correct and it implies that he was not focusing only on the bases and their exponents, as Zacher believed. In contemporary terms, the determination of the digits of a number like $101 = (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0)$ never depends strictly on the base, which is always a 100, 10, 1 multiplier, but on the other factor of the products, which can either be 1 or 0. Since in his binary algebra Leibniz assigns to the symbols a value that can also be 0, it means that he was considering also the other factor in the products.

$$(A) \ 14^2 = 196 \rightarrow (1 + 4)^2 = 1 + 9 + 6$$

<i>III</i>	<i>II</i>	<i>I</i>	
12^2		11^2	
	$\textcircled{2}4 \times 1$	4×4	
1×1	$+1(c)$		$= b^2(1)$
1	9	6	$= b^2(3)$

$$(B) \ 11^2 = 1001 \rightarrow (1 + 1)^2 = 1 + 0 + 0 + 1$$

<i>IV</i>	<i>III</i>	<i>II</i>	<i>I</i>	
	12^2		11^2	
$+1(c)$	1×1	0	1	
	$+1$			$= b^2(2)$
1	0	0	1	$= b^2(3)$

Apart from the carry, indicated with the symbol (c) next to it, that could appear in certain circumstances when a number exceeds its base in the previous column, these examples follow strictly the algorithms developed by Leibniz, even if the example for (B) might be harder to follow: recall we are in base 2, so that the number 2×1 (that would be written with a circle on the 2 in algorithm (A)) should be rewritten as 10 and corresponds to adding a zero to the next column on the left.

Through these artifices, not only Leibniz is able to express the square of a binary series eliminating all the powers and all the double-products involved, but he can also effectively highlight relations between any possible series, including series representing irrational numbers, provided these are expressed in the binary numeral system. For instance, we now know that for any binary number, its first digit is always equal to the first digit of its square (21 is in the same position of 11 in the general table presented above) and the second digit will always be equal to 0.¹⁷ We also know for example, that the number

17. This is the reason why in his original table Leibniz, causing even more confusion for the reader, does not assign a symbol for the second place of the square's series: it will always be 0, hence a known quantity.

displayed at the sixth position of the squared series (26) is equal to the sum between the product of the numbers at the first and fifth position and the product of the numbers at the second and fourth position of the root series: $26 = (11, 15) + (12, 14)$, provided again that we maintain the correct exponent for the base. Depending on the number, these products and sums might be equal to 0, yet this kind of knowledge will always be more than what we can infer from the correspondent decimal series and, having the values of some of these numbers at our disposal, we can find the value of the remaining ones, following the table. The power of Leibniz's binary algebra, where these symbols can only represent ones or zeros, is that we can make an educated guess on future calculations because there can only be two hypotheses and the numbers involved in them have peculiar properties when multiplied or added.

This binary algorithm was an original mathematical achievement in Leibniz's eyes and one of the greatest proofs of the superiority of the binary numeral system, so much so that he himself calls it a *fastigium*, giving it a wide scope that involves the expression of any possible number:

With this method it will be possible to find the progression of the binary characters expressing the value $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, as well as $\sqrt[3]{2}$, $\sqrt[3]{4}$, $\sqrt[3]{5}$, etc., or of any other pure simple radix. In the same way both composed radices and universal radices will be found, even if imaginary [numbers] would be included, as in $\sqrt{(1 + \sqrt{-2})} + \sqrt{(1 - \sqrt{-2})}$. [Zacher 1973, 222]

While the actual result might look underwhelming for now compared to these claims, because it concerns only the study of a series and its square, the idea that Leibniz suggests is that the more we study the possible relationships between general infinite series in his positional notation, the more we understand the behaviour behind the development of numbers' digits, even irrational numbers' infinite digits. This grand project was never shown completely by Leibniz, but a glimpse into the possibility of, in a manner of speaking, rationalising the irrationals was more than enough to establish also at a mathematical level the priority of the binary numeral system over any other possible system.

4 Irrational and transcendent quantities: the study on numbers' period

Despite the progress obtained thanks to his algorithm, Leibniz is perfectly aware of the discrepancy between ideal results and real results, when the number expressed in the positional notation is an irrational number. Following Leibniz's notion of possibility, a number having digits prolonged to infinity without presenting a period is a concept that doesn't lead to a contradiction, thus conceivable, but at the same time, there cannot be real instances of such

a number, because any expression of its value would be only an approximation. Binary algebra works for irrational numbers only in general, when applied to symbols and not actual digits, if imagining a continuation to infinity of the algorithms describing them is possible, but expressing the actual values in the positional notation is problematic. Leibniz's first solution appeals to the homogeneity in the expression sought by his binary algebra: since the new notation and its properties can be applied to any number, it can be applied also to the rational numbers used as approximations of the irrational value. By comparing different approximations then, we should be able to understand the behaviour of the infinite series grounding those results:

But in infinite series, if we want to find the rule, first we calculate quasi- $b = 11$, then quasi- $b = 11 + 12$, then again quasi- $b = 11 + 12 + 13$, and so forth until the progression appears. [Zacher 1973, 219]

If trying to compare rational and irrational numbers is still understandable in the context of 17th-century mathematics, what remains puzzling in the manuscripts on dyadics is Leibniz's use of the notion of transcendence:¹⁸

Finally, since there does not exist a transcendent quantity which cannot be expressed through numbers prolonged in infinity [meaning: indefinitely], in any case, it will always be possible to find some kind of rule of the progression. [Zacher 1973, 218]

Since in this passage Leibniz seems to approach transcendent quantities in present-day terms, this would advocate against well-established interpretations by other scholars,¹⁹ who see Leibniz indeed as the inventor of the term used nowadays, but not also as an anticipator of the present notion, especially concerning the classification of numbers, because the idea itself of a transcendental number is scarcely present in his works. For the purpose of this paper, since transcendental numbers like π are also irrational numbers and since Leibniz's method for the binary expression of digits is also applicable to them, there would be no need to look for an actual distinction, but a deeper knowledge of dyadics' manuscripts suggests otherwise: in many of them, including *De progressionem dyadica*, transcendent quantities are thematised separately and they are the reason why Leibniz tries to find an alternate solution to that of approximation to deal with infinite digits.

In fact, the first solution didn't convey very well the novelty and importance of Leibniz's positional notation and this is why for his whole life he tried to achieve the same result in another way, connected to another fundamental discovery on the binary numeral system: the period of natural

18. I'm using the terms "transcendent number" or "transcendent quantity" to indicate Leibniz's notion of transcendence, and the terms "transcendental number" to indicate the present notion.

19. [Bourbaki 1984, 98] and more recently [Serfati 2018, 85].

numbers. From the very beginning of his studies on dyadics, Leibniz discovered some interesting properties concerning series of numbers written in their binary form:²⁰

Binary								Decimal
0	0	0	0	0	0	0		0
0	0	0	0	0	0	1		1
0	0	0	0	0	1	0		2
0	0	0	0	0	1	1		3
0	0	0	0	1	0	0		4
0	0	0	0	1	0	1		5
0	0	0	0	1	1	0		6
0	0	0	0	1	1	1		7
0	0	0	1	0	0	0		8
0	0	0	1	0	0	1		9
0	0	0	1	0	1	0		10

Table 1 – Natural numbers

Binary								Decimal
0	0	0	0	0	0	1		1
0	0	0	0	0	1	1		3
0	0	0	0	1	0	1		5
0	0	0	0	1	1	1		7
0	0	0	1	0	0	1		9
0	0	0	1	0	1	1		11
0	0	0	1	1	0	1		13
0	0	0	1	1	1	1		15
0	0	1	0	0	0	1		17
0	0	1	0	0	1	1		19

Table 2 – Odd positive integers

20. The following tables are present throughout Leibniz's whole production, from the *Summum calculi analytici fastigium* [Zacher 1973, 223], to Leibniz's letter to Bouvet [Zacher 1973, 243] or the *Essay d'une nouvelle science des nombres* [Zacher 1973, 261], and many more.

Reading binary natural numbers in the vertical sense in the table (1), Leibniz notices that the digits of every number are not only composed by values that are either 1 or 0, but they are also part of a period of ones and zeros: 01 for the first digits, 0011 for the second digits, 00001111 for the third digits, and so forth. Much like the algorithms for the squares previously developed, this information was seen by Leibniz as a great contribution to his positional notation and he tried to implement it, applying again his binary algebra.

The best expression of this implementation is contained in the *Periodus numerorum* [LHXXXV, 12, 1, fol. 190–191], an unpublished manuscript written by Overbeck for Leibniz and dated probably around 1705, definitely subsequent to the ones written in 1679. To convey the implementation of the periods, Leibniz adopts a notation that shows the number of ones and zeros contained in each of them, comparing it then to his binary algebra's notation:²¹

$$\begin{array}{cccccccccccccccc} 0_{256}1_{256} & + & 0_{128}1_{128} & + & 0_{64}1_{64} & + & 0_{32}1_{32} & + & 0_{16}1_{16} & + & 0_81_8 & + & 0_41_4 & + & 0_21_2 & + & 0_11_1 & = & N \\ 18 & & + & 17 & & + & 16 & & + & 15 & & + & 14 & & + & 13 & & + & 12 & & + & 11 & & + & 10 & = & N \end{array}$$

The new notation is extremely interesting, because it expresses all the possible natural numbers in one string, in a way that was accessible to Leibniz only at that time. For every series, the subscript indicates the number of ones and zeros repeated and the algebraization allows operations between the series. In the *Periodus numerorum* in fact, Leibniz does not describe only the algorithms already found in the *Summum calculi analytici fastigium*, but shows also how to operate between the periodic series, introducing the operations of addition and multiplication and their generalisation. Leibniz wanted to combine his positional notation previously developed and the property possessed by every number as being part of the tables now introduced. For example, the binary number 101 will be expressed in its positional notation as $1+0+1$ and its algebraic expression will be $12+11+10$ (in this manuscript, the first algebraic number is 10 and not 11, like in the previous ones). At the same time however, the first digit of this number is also a 1 that belongs to the period 01 in the first column of the table (1), the second digit is a 0 belonging to the period 0011 in the second column and the third digit is a 1 belonging to the period 00001111 of the third column, expressed by Leibniz respectively with $0_41_4 + 0_21_2 + 0_11_1$. While in this newly discovered form we might not know whether the number expressed is precisely 1 or 0, much like in the original binary algebra, the fact that it belongs to a new, wider and comprehensive way of expressing any possible number is clear, and it might be of use to obtain

21. For space reasons, the last digit from the left written by Leibniz ($0_{512}1_{512}$) is missing. Also in the original, the + sign is conveniently substituted by vertical lines in order to highlight the correspondence between the new notation and the binary algebraic numbers [see LHXXXV, 12, 1, fol. 190r] and the numbers written as subscripts are conceived on the same type line of the ones and the zeroes, the latter however being larger.

information on numbers in certain contests in which other information are given, hence the combinatorial approach with the simple positional notation.

This result explains why in the early manuscripts on dyadics there is always a section on transcendent quantities, specifically the area of a circle having the unity as a diameter. After Gregory, also Leibniz discovers that the quantity $\frac{\pi}{4}$ can be expressed as the alternating series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \dots$. This series was already studied by many scholars for its role in the problem of squaring the circle and for its meaning in the expression of numbers, since Leibniz states that, even if the number designated is not finite, we can have an exact knowledge of it, because we have access to the law generalising the series, in modern notation: $\frac{(-1)^k}{2k+1}$.²² In the manuscripts on dyadics, a different solution is suggested: the idea is to manipulate the series in a way that allows for an easier expression of it. Above all, even if the series is an alternate series, Leibniz ties the manipulation to the study of the series of odd positive integers, $1+3+5+7+9\dots$, which can be derived in some way from the original series. Having presented Leibniz's achievements on dyadics, the reason is now evident: comparing table (1) and table (2) shown above, we can clearly see that the periods generated by the odd positive integers is the same as the one generated by the natural numbers, except shifted by one position and having at the first place the period 1. Leibniz's aim then is trying to find the positional notation for $\frac{\pi}{4}$ or π , deriving it from the positional notation of the odd series,²³ which is at the same time derived from the positional notation of the natural series. This project was only sketched in the early writings because Leibniz had not yet developed the correct tools needed to treat the complex positional notation, but if we were able to use the results obtained in the *Periodus numerorum*, we could say that the odd series can be expressed in his positional notation as $N+1$. The fact that Leibniz spent his entire life trying to obtain this result is the testament of how important and effective he considered the binary notation.

The connection between dyadics and transcendent quantities shows then that Leibniz was much closer to today's notion of transcendental number than it is generally believed. In *De progressionem dyadica*, we find a definition of a transcendent quantity usually ignored by the readers:

It is demonstrated then that transcendent quantities are given, i.e., those that cannot be expressed with an equation having definite exponents. [LH XXXV, 3, B 2, fol. 4]

Leibniz surely has in mind the alternate series, but he also has in mind his just developed positional notation. Indeed, if we compare today's formula of a polynomial equation and that of the expression of every positive integer, some

²² On the general problem of squaring the circle, see [Crippa 2019]. On the role of the series for the concept of expression, see [Debuiche 2013, 422–423].

²³ On this topic see also [Shukhman & Shukhman 2017].

similarities are definitely present:

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0.$$

Where $n \geq 1$ and a_i are rational numbers

$$r_m b^m + r_{m-1} b^{m-1} + \dots + r_1 b + r_0 = a.$$

Where $0 < r_m < b$ and r_i are positive integers.

In the first formula, x is the unknown quantity, and transcendental numbers are in some way connected to it because they cannot be in any case roots of a nonzero polynomial equation of this kind, i.e., they are not algebraic numbers. In the second formula b is instead a known quantity, but for transcendent quantities, as Leibniz instead intended them, it is the exponent of the base which is unknown (m), and specifically, it is indeterminable because it is prolonged indefinitely. The similarities would end here, if we were to consider Leibniz's expression of every positive integer as a common sum of an infinite power series, because the transcendence of the number considered wouldn't follow from how it is expressed and we would have to also prove separately that, apart from being expressed in this way, a finite expression is not possible. At the same time however, Leibniz possesses the positional notation able to express these numbers, showing algorithms and recurring properties: much like $\frac{(-1)^k}{2k+1}$, the general law for the expression of $\frac{\pi}{4}$ or π can be studied, this time using the binary system.

What I am suggesting then is that Leibniz's attempts to turn the infinite series expressing $\frac{\pi}{4}$ or π into something that involves the use of the series of odd positive integers are likewise attempts to express $\frac{\pi}{4}$ or π , or π as positional binary notation series, i.e., series having a fixed order. A parallelism with natural numbers can be easily drawn: every natural number (e.g., 49) can be expressed as the sum of many other positive numbers ($40 + 9$ or $23 + 26 \dots$), but there can only be one positional expression as a sum ($4 \times 10^1 + 9 \times 10^0$). In this sense, for transcendental numbers, the obtained series wouldn't be an ordinary series, but the one from which all the others derive because it is the series that turns a seemingly ordinary sum into the formula for the expression of every positive integer in a fixed order, or into a formula that can be directly derived from it in some way. If this transformation was possible, there would not be the need of also proving that the number considered cannot be expressed through a finite expression, because its infinite expression would also be the expression of its actual digits. There are indeed convergent infinite series that can be expressed through a finite number, but asking the same for a positional infinite series would be the same as asking to express the same number with a different number of digits or with different values for its digits.

While this might happen in some cases,²⁴ Leibniz believed that bridging the gap between $\frac{\pi}{4}$ and its expression through the series of odd numbers would have probably illuminated the path towards a better understanding of transcendent quantities, much as dyadics already did for a quantity and its square.

At the same time, this would have meant being able to express quantities like π in a positional notation showing their actual digits and not their approximations, something that we know is not possible. Leibniz then was experimenting with this idea without ever achieving a definitive result, guided by his belief that the binary expression of a series is a superior form of expression, but this remark can only be the starting point of a wider study on the relationship between the binary numeral system and the expression of transcendent quantities. It shows however that Leibniz's efforts on dyadics were strongly connected with his mathematical efforts started before 1679.

5 Conclusion

Other than being temporally subsequent, *De progressionem dyadica*, the *Summum calculi analytici fastigium* and the *Periodus numerorum* show clearly the evolution of Leibniz's dyadics: from the use of binary algebra with common letters for the study of finite quantities to the introduction of a better notation for the expression of any possible number and the periods of natural numbers; from the description of transcendent quantities sketched in the early writings to its mature form in the new positional notation.

A renewed interest in Leibniz's contribution to the binary numeral system has already shed some light on the connection with its further developments.²⁵ The mathematical results presented in this paper, e.g., the positional expression of two algorithms describing the expansion of the squares or the expressions and operations between infinite periodic series, show a similar trend because they are all connected to what I defined as Leibniz's binary algebra. They are particularly important also because they depend on notions accessible only to Leibniz at that time. The transition between *De progressionem dyadica*'s first and second part, that is the transition between binary arithmetic and binary algebra, was then for Leibniz almost more important than the discovery of the binary numeral system itself:

I approach now its Algebra, which I surely intend in a different way than the ordinary notion, since I will assume the variables not as unknown quantities, but in place of characters designating the numbers requested, expressed in Dyadic form. Something which

24. For example, it is the case of $0.999\dots = 1$, which is not by chance another main topic in Leibniz's manuscripts on dyadics. It also happens when we express the same number in another number base, but this case does not count as a counterexample in this context.

25. See for example [Arthur 2020].

until now no one did. Marvellously moreover, with this artifice all things become related. [LH XXXV, 3, B 2, fol. 2]

Bibliography

- ARTHUR, Richard [2020], Leibniz as a precursor to Chaitin's algorithmic information theory, in: *Information and the History of Philosophy*, edited by Ch. Meyns, London: Routledge, forthcoming.
- BERGGREN, Lennart, BORWEIN, Jonathan, *et al.* [2004], *Pi: A Source Book*, New York: Springer, doi: 10.1007/978-1-4757-2736-4.
- BOURBAKI, Nicolas [1984], *Éléments d'histoire des mathématiques*, Paris: Masson.
- BRANCATO, Mattia [2016], Leibniz, Weigel and the birth of binary arithmetic, *Lexicon Philosophicum*, 4, 151–172.
- CRIPPA, Davide [2019], *The Impossibility of Squaring the Circle in the 17th Century. A debate among Gregory, Huygens and Leibniz*, Basel: Birkhäuser.
- DEBUICHE, Valérie [2013], L'expression leibnizienne et ses modèles mathématiques, *Journal of the History of Philosophy*, 51(3), 409–459, doi: 10.1353/hph.2013.0049.
- [2019], Expression and analogy in Leibniz's philosophy, in: *Äußerungen des Inneren*, edited by L. E. Herrera, Berlin; Boston: De Gruyter, 65–84, doi: 10.1515/9783110666120-004.
- GLASER, Anton [1971], *History of Binary and Other Nondecimal Numeration*, Los Angeles: Tomash Publisher.
- KNOBLOCH, Eberhard [2010], Leibniz between *ars characteristica* and *ars inveniendi* : Unknown news about Cajori's “master-builder” of mathematical notations, in: *Philosophical Aspects of Symbolic Reasoning in Early Modern Mathematics*, edited by A. Heeffer & M. van Dyck, London: College Publications, Studies in Logic, 289–302.
- LEIBNIZ, Gottfried Wilhelm [1903], *Opusculs et fragments inédits*, Paris: Félix Alcan, edited by L. Couturat.
- [1966], De Progressione Dyadica, Pars I, (MS, 15 March 1679), in: *Herrn von Leibniz' Rechnung mit Null und Eins*, edited by E. Hochstetter & H.-J. Greve, Berlin: Siemens Aktiengesellschaft, 42–47, Available also as Leibniz, G. W. [1971], Leibniz. *Calcolo con zero e uno*, edited by L. Ruffini, Milan: Etas Kompass.

- [1976], *Ein Dialog zur Einführung in die Arithmetik und Algebra, nach der Originalhandschrift herausgegeben, übersetzt und kommentiert*, Stuttgart: Frommann-Holzboog, edited by E. Knobloch.
- [2018], *Mathesis universalis. Écrits sur la mathématique universelle*, Paris: Vrin, textes introduits, traduits et annotés sous la direction de D. Rabouin.
- MAITRE, Marie-Julie [2020], *L'Analogie de Joachim Bouvet entre le Xiantiantu de Shao Yong et l'arithmétique binaire de Leibniz. Analyse d'une rencontre intellectuelle transculturelle*, New Taipei City: Sili danjiang daxue faguoyu wenxue xi.
- NIKOLIC, Aleksandar [1994], *Gottfried Wilhelm Leibniz et le système binaire, Faculty of Science, University of Novi Sad. Mathematics series*, vol. 24, Novi Sad Fac. of Science.
- OSLER, J., Thomas [2010], The tables of John Wallis and the discovery of his product for Π , *The Mathematical Gazette*, 94(531), 430–437, doi: 10.2307/25759727.
- PROBST, Siegmund [2015], Leibniz as reader and second inventor: The cases of Barrow and Mengoli, in: *G. W. Leibniz, Interrelations between Mathematics and Philosophy*, edited by N. B. Goethe, Ph. Beeley, & D. Rabouin, Dordrecht: Springer, 111–134, doi: 10.1007/978-94-017-9664-4_6.
- [2018], The relation between Leibniz and Wallis: an overview from new sources and studies, in: *Quaderns d'Història de l'Enginyeria*, vol. XVI, 189–208.
- SERFATI, Michel [2018], *Leibniz and the Invention of Mathematical Transcendence*, Stuttgart: Franz Steiner Verlag.
- SHUKHMAN, Elena V. & SHUKHMAN, A. E. [2017], Unpublished manuscripts by g. w. leibniz associated with nondecimal systems, in: *ACMPT-2017*, 65–69, URL http://acmpt.ru/wp-content/uploads/2017/06/ACMPT_2017_paper_113.pdf.
- WALLIS, John [1657], *Mathesis universalis sive arithmeticon opus integrum tum Numerosam. Arithmeticon tum Speciosum complectens*, Oxford: Lichfield.
- ZACHER, Hans J. [1973], *Die Hauptschriften zur Dyadik von G. W. Leibniz*, Frankfurt am Main: Vittorio Klostermann Verlag.