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Analysis, si uti scias, potens est. Reappraisal of Heuristic Power of Greek Geometrical Analysis

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Analysis, si uti scias, potens est Reappraisal of Heuristic Power of Greek Geometrical Analysis

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Résumé : Dans cet article, nous évaluons la puissance heuristique de l'analyse géométrique grecque en essayant de reconstruire l'analyse de certaines propositions dont la démonstration (synthèse) est encore existante. Nous reconstruisons notamment l'analyse de (1) la trisection de l'angle (exposée par Pappus dans le Livre IV de sa *Collection*), (2) la propriété de la tangente à la parabole, (3) à l'hyperbole/ellipse (Apollonius, *Coniques* Livre I) et (4) à la spirale (Archimède, *Sur les spirales*). Dans tous ces cas, les résultats et les démonstrations peuvent être trouvés par l'analyse seule, sans arguments par analogie avec d'autres figures, et sans considérations cinématiques. Dans ces cas, l'analyse se révèle être un outil beaucoup plus puissant qu'on ne le suppose habituellement. Nous examinons également une partie du rapport *Section selon le rapport* d'Apollonius, pour montrer que le nombre de possibles cas à distinguer dans l'analyse tendait à augmenter, puisque les objets géométriques et les magnitudes qui y sont liées étaient toujours pris en compte dans la disposition géométrique dans laquelle ils apparaissent. En effet, l'absence d'une approche algébrique, qui aurait détaché les magnitudes de leur contexte géométrique, a été l'un des principaux obstacles au développement de la géométrie grecque.

Abstract: In this article, we assess the heuristic power of Greek geometrical analysis by trying to reconstruct some analyses of extant propositions of which only the demonstration (synthesis) is found in the text. We have reconstructed the analysis of (1) the trisection of an angle (Pappus' *Collection*, Book IV), (2) the property of the tangent to the parabola, (3) to the hyperbola/ellipse (Apollonius' *Conics*, Book I), and (4) to the spiral line (Archimedes' *On Spirals*). In all of these cases, the results and the demonstrations can be found by the analysis alone, without arguments by analogy with other figures or kinematic considerations. We have thus concluded that analysis is a much more powerful tool than is usually assumed to be. We have also examined a

part of Apollonius' *Cutting off of a ratio* and confirmed that the number of possible cases to be distinguished in analysis tended to increase because the geometric objects and the magnitudes related to them were always considered in the geometric arrangement in which they appear. We suggest that the absence of an algebraic approach that would have detached the magnitudes from their geometrical context may have been one of the major obstacles to further development of Greek geometry.

1 Introduction

This article examines how Greek geometrical analysis was used, and tries to evaluate to what extent it was useful as a tool for investigations in geometry.¹ As its title suggests, the conclusion is that the analysis was a much more

1. For the readers who are not specialists of Greek mathematics, we briefly explain Greek geometrical analysis, and a much debated issue concerning it, which however we do not treat in this article.

Analysis in ancient Greek mathematics is a research approach which consists in assuming the required result as if it were obtained. This is a simple trick that everyone who learns elementary geometry uses to solve a problem even without being taught to do so. The particular style and terminology of the ancient Greek analysis, with a typical example, is best illustrated in [Behboud 1994].

The central issue, since the fundamental study [Hintikka & Remes 1974], is the “direction” of the analysis. However, we consider that this was a problem invented by modern scholarship, and had little to do with Greek geometry itself. The problem arose from Pappus’ rather vague and inconsistent presentation of analysis at the very beginning of Book VII of the *Collection*. In this book, Pappus enumerates the works which he thought were useful to learn analysis, and collects about 150 propositions which would serve to understand the propositions in those works.

Pappus’ definition of analysis begins as follows:

Now, analysis is the path from what one is seeking, as if it were established, by way of its consequences, to something that is established by synthesis. That is to say, in analysis we assume what is sought as if it has been achieved, and look for the thing from which it follows, and again what comes before that, until by regressing in this way we come upon some one of the things that are already known, [...].

(Translation in [Jones 1986, 1:82])

The interest of the historians is concentrated on the direction of the reasoning in analysis. Let us assume that the figure to be constructed has been obtained, or the property to be demonstrated is true (let us call it assumption A). Then, one finds B logically related to it. But is B one of the logical consequence that follows from A , or is it one of the properties from which A can be derived? That is, which did Pappus mean, (1) or (2)?

- (1) $A \leftarrow B \leftarrow C \dots$
- (2) $A \rightarrow B \rightarrow C \dots$

powerful tool than usually supposed. We also argue that analysis can also find new theorems. If the power of the analysis is not duly evaluated, this is mainly due to the inexpertness in Greek geometry on the part of modern historians. There are a number of reasons for this inexpertness. We know more advanced mathematics which can resolve all the problems that the Greeks solved, and even many they could not solve, so we are not motivated to dedicate as much effort to learn Greek analysis as ancient mathematicians. We can add that today there are no teachers who can teach us Greek geometry and show how to use their analysis.

In this article, we try to teach ourselves the method of analysis by reconstructing possible analyses from the demonstrations of complicated theorems.² We start with a solution to the problem of trisecting an angle in Pappus' Book IV, and reconstruct its analysis. This rather simple exercise shows that geometrical objects were not like quantities in algebraic expression. Each object was particular, in the sense that it has its own position and spatial relation to other objects, and particular techniques or "tips" were necessary to treat such objects.

More complicated propositions are then examined: determination of tangents to the conic sections (in Apollonius' *Conics*) and to the spiral line (in Archimedes' *On Spirals*). In the demonstrations of these propositions, the authors give no suggestion about how they have found the result they prove. A reconstructed analysis is proposed for each of them, which (hopefully) will appear to be possible to find even without knowing the result beforehand. The tangent to the parabola (*Conics* I.33) is treated in the main text of the present article, while more complicated propositions are examined in the appendix: tangent to the ellipse/hyperbola (*Conics* I.34), and to the spiral

Pappus' words cited above seem to suggest (1), but some of his words after the quoted passage can be interpreted to mean (2).

However, this is not as important a difference as it seems, for most of the deductions in mathematics are reversible: for example, in isosceles triangles the angles at the base are equal (*Elements* I.5), and if two angles in a triangle are equal, the two sides subtending the equal angles are equal, so that the triangle is isosceles (*Elements* I.6). Indeed, before [Hintikka & Remes 1974], Mahoney had already pointed out this reversibility of mathematical deduction, making much less important the question about the direction of deduction in analysis [Mahoney 1968]. More recently, Acerbi, by careful examination of each term used by Pappus, convincingly argued that the alleged inconsistency of Pappus' presentation does not really exist [Euclide 2007, 442–445].

The real problem in constructing synthesis (ordinary demonstration) by inversion of the accomplished analysis, is the existence of the solution. Some problems do not always have solution, and it is necessary to determine the initial conditions which allow one or more solutions. Such arguments are called diorism, which occupies not a small part of analysis [see Saito & Sidoli 2012].

2. The approach of this article was pursued in [Knorr 1986] which is still one of the most important and illuminating works in the history of Greek mathematics. We try to show the power of analysis more clearly, by careful examination of limited number of complicated propositions in extant works.

(*On Spirals*, prop. 18, and its lemmata prop. 7 and 8). The readers are invited to examine the appendix as well, in order to appreciate the power of Greek geometrical analysis.

Before the conclusion, the limitations of Greek geometric analysis are discussed. It is true that the reach of geometric analysis was quite limited. The limitation is mainly due to the nature of the objects of Greek geometry. They were figures, and were treated where they are and as they are, and no abstract operation was possible. Quantities, which we treat in algebra, have no position, and we apply algebraic formulas and transformations to such quantities, which are objects abstracted from geometrical figures, depriving them of their geometrical particularity such as position and positional relationships to other objects. This is a process alien to Greek geometry. The objects of Greek geometry were always accompanied by their position and configuration, and this entailed frequent and tedious division of cases according to the arrangement of the figure. We will show an example of explosive multiplication of cases in Apollonius' *Sectio rationis* (*Cutting off of a ratio*).

2 Reconstructed analyses

2.1 Angle trisection in Pappus IV.32

This proposition treats the trisection of an angle, and the problem is reduced to another one, that of *neusis* (inclination), which we will explain in the course of the presentation of the analysis.³

Figure 1: Angle trisection, reconstructed analysis (1) [Pappus 1876-1878, 1:274ff.]

Let an angle ABC be given (Fig. 1). And it is proposed to divide it into three equal parts. Let it be done (a typical beginning of an analysis), and let

3. In the preceding proposition IV.31, Pappus resolves this *neusis* problem using an hyperbola. So Pappus did not consider the solution by *neusis* as a legitimate means of construction, and it had to be further reduced to another solution. We here concentrate on the discovery of the solution by *neusis*.

the straight line BE be the trisection of the angle ABC. And let the line AE be drawn parallel to BC. Then the line AE is given, and we show it by solid line, while the line BE by a dashed line, for this line is not yet given, i.e., we don't know where it is. So the angle AEB is equal to the angle EBC, so that the angle ABE is double the angle AEB. This means, although we don't know either of the angles, we know that the former is twice the latter.

Now, let the line AH be drawn, which makes the angle EAH equal to the angle AEH (Fig. 2). Why can we suppose that a Greek mathematician could have the idea of drawing AH? With this line one would have an isosceles triangle AEH, and its exterior angle AHB is double of the angle AEB, that is, equal to the angle ABH. Therefore, the triangle ABH is isosceles, so AB is equal to AH, which is also equal to HE. As AB is given in position and in magnitude, both AH and HE are given in magnitude (but not yet in position).⁴

Figure 2: Angle trisection, reconstructed analysis (2)

This is significant progress. Now, draw AC, perpendicular to BC (so AC is given), and let AC, BE meet at point D (Fig. 3). Then AHD is another isosceles triangle, for both the angles HAD and HDA are one right angle less the angle AED. Therefore, DH is equal to AH, and since AH is given in magnitude, so is DH. Therefore, DE (= DH + HE = 2DH) is also given in magnitude. And the lines AE and AC are given in position.

So the problem is reduced to another one, to set a line DE, equal to double of AB, between two lines AE and AC, verging toward the point B. This sort of configuration is called a *neusis* (inclination) in Greek. We restate it more faithfully to Pappus' text.

Between the given lines CAE, let a line ED be put (χείσθω)
inflected (νεύουσα) toward (ἐπὶ τὸ) B, equal to twice AB. [Pappus
1876-1878, 1:276]

4. To be able to go ahead, it is required to “see” the triangle AHB which is not yet drawn in the diagram. To be able to “see” it, one has to know, not only that the angles at the base of an isosceles triangle are equal to each other (*Elements* I.5), but also that the exterior angle at the vertex of an isosceles triangle is double of the angle at the base. The latter is an obvious corollary of *Elements* I.5, but one has to keep it in memory in its explicit form. We will come back to this argument later in §2.2.

Figure 3: Angle trisection, reconstructed analysis (3)

2.2 Tool-box

This example shows us not only how the analysis works, but also why it is not easy to use analysis. The key step is to draw the line AH , which makes two isosceles triangles AEH and ABH . Without this step the analysis gets nowhere. Then, how could a Greek mathematician find this construction?

Let us go back to the figure before drawing the auxiliary line AH (Fig. 1). We have the angles EBC and AEB , one-third of the given angle, and the angle ABE , which is double of the angle AEB . Then, the task at this stage is to somehow relate the angle AEB with its double, the angle ABE .

For us, to double something we call x is to take $2x$. But this makes no sense in geometry. Instead, there were several ways to double a given geometrical object, depending on what that object is, and on the kind of configuration in which it is found. For example, if it is an angle on the circumference of a circle, one can get its double by constructing the angle at the center on the same arc or an equal arc. In this case, we have to remember another property. Namely, the exterior angle at the vertex of an isosceles triangle is double the angle at the base. So it is natural to try constructing the isosceles triangle AEH .

An ancient geometer knew various specific ways of doubling a geometrical object, and when it comes to an angle, one could at once think of the angles in a circle, or the exterior angle of an isosceles triangle. Put in another way, there were several different tools to double a geometric object, according to whether it was an angle, a line, or a surface. The process was not something like writing $2x$ instead of x . For us the expression $2x$ makes sense, for we have all the algebraic formulas, theorems and functions—for example, trigonometric functions help us to get algebraic expression for various geometric configurations. The construction of the regular pentagon is reduced to the calculation of the value of sine or cosine of 72 degrees. In Antiquity, in the absence of such a set of tools, the situation was completely different. The expression like $2x$ did not make sense. Thus, it was not enough to know that the exterior angle of a triangle is equal to the sum of the opposite internal

angles (*Elements* I.32), but one also had to know that this property could be used to double an angle by constructing an isosceles triangle.⁵

If the demonstrations in Greek mathematical texts look like an aggregate of nice but inexplicable inspirations, and if we cannot but suspect that Greek mathematicians hid their true techniques as Descartes believed, it is because we have not learned Greek geometry in the way that the ancients did. One has to know how to use tools. If one has never seen a screw, one may take it for a nail, and perhaps use a hammer, not a screwdriver, to drive it into a hole. Various techniques for doubling an angle in Greek mathematics can be compared to screwdrivers. They had cross head (Phillips) screwdrivers and slotted head screwdrivers, and used them according to the screw they came across. We moderns have the golden hammer called algebra, which we can use everywhere. So the moral we should remember when we read Greek mathematics is: when all you have is a hammer, everything looks like a nail (Maslow's law).

2.3 Tangent to the parabola (*Conics* I.33)

Let us turn again to an ancient source and see another example which shows the power of analysis: the determination of the tangent to a parabola (*Conics* I.33).⁶ In Fig. 4, the curve ΓE is a parabola, whose diameter is AB . Γ is a point on parabola, and $\Gamma\Delta$ is the ordinate drawn from Γ . Apollonius shows that if AE is equal to $E\Delta$, then ΓA is tangent to the parabola, without explaining how he got the idea of making AE equal to $E\Delta$.

To reconstruct the analysis, let us start from Apollonius' conclusion, that the line ΓA is tangent. Then, any point Z on ΓA falls outside of the parabola.⁷ Following Apollonius' proof in reverse order, one should arrive at the relation that AE is equal to $E\Delta$.

5. The set of useful propositions (or tools) in the investigation of Greek mathematicians is expressed by the term "tool-box" [Netz 1999b, 216ff.], [Chemla 2012, 30].

6. We use [Apollonius 2008b] for the text of the *Conics*. Since all the extant manuscripts of the *Conics* derive from Vat.Gr. 206, the recent critical edition [Apollonius 2008b] (with French translation) does not present significant differences. The diagrams in the latter are very different from those in the manuscripts (especially in I.34 which is treated in appendix of this article) but metrically correct, while those in Heiberg's edition [Apollonius 1891-1893] are a little bit closer to the manuscript diagrams.

A critical edition (with French translation) of the Arabic version of the *Conics* is also available [Apollonius 2008a].

7. Apollonius formulates his argument as a proof by contradiction, and assumes that a point Z on AE falls inside the parabola and proves that this is impossible. In the diagram also, the point Z is inside the parabola. However, the mathematical arguments do not essentially change if we reconstruct an analysis departing from the premise that Z is outside the parabola as we do here. We have also changed the diagram.

Figure 4: *Conics* I.33. Tangent to a parabola.

On the tangent ΓA , let a point Z be taken, and let ZHB be drawn, parallel to the ordinate $\Gamma\Delta$. Since ΓA is tangent to the curve, Z must fall outside the parabola (Z can also be between A and Γ), that is

$$ZB > HB. \quad (1)$$

Since $\Gamma\Delta$ and HB are ordinates, the squares on them are as the abscissas, that is

$$q(\Gamma\Delta) : q(HB) = E\Delta : EB. \quad (2)$$

And since $\Gamma\Delta$ and ZB are parallel, triangles $A\Gamma\Delta$ and AZB are similar, so

$$q(\Gamma\Delta) : q(ZB) = q(A\Delta) : q(AB). \quad (3)$$

From (1), (2) and (3),

$$E\Delta : EB > q(A\Delta) : q(AB). \quad (4)$$

This relation is not easy to handle, for the left side is a ratio of lines while the right side is that of areas. So let us add some line to the left side, and convert the ratio on the left side to a ratio between rectangles.

Since a ratio is easier to handle if its terms contain more common lines and common points, it is a good idea to choose AE as the side of the rectangle. Thus we have

$$r(AE, E\Delta) : r(AE, EB) > q(A\Delta) : q(AB). \quad (5)$$

Exchanging the inner terms, we have

$$r(AE, E\Delta) : q(A\Delta) > r(AE, EB) : q(AB). \quad (6)$$

Why is one motivated to exchange the inner terms? Because only three points appear in each of the ratios of both sides after exchanging the inner terms. It is easier to handle.

Now, the ratios on the left and right sides are those of the rectangles contained by two segments of a line to the square of the whole line. Every Greek mathematician knew that such a rectangle is largest when the lines are divided into two equal parts, making a square on the half of the whole line. This is a straightforward result of *Elements* II.5.

Therefore, on either side, four times the antecedent (four times the rectangle) is equal to or less than the consequent. So the relation is more directly visible if one takes four times the antecedents:

$$4r(\text{AE}, \text{E}\Delta) : q(\text{A}\Delta) > 4r(\text{AE}, \text{EB}) : q(\text{AB}). \quad (7)$$

Then, on both sides of (7), the antecedent is equal to the consequent only when E is the midpoint of the line AΔ (left side), or of AB (right side).

So, if E is *not* the midpoint of AΔ, taking the point B so that AE is equal to EB, one can make the ratio on the right side bigger than that of the left side, which is a contradiction. Therefore, E must be the midpoint of AΔ.

Modern scholars wonder how Apollonius found this proposition. However, the transition from (4) to (5) is not so mysterious as it may seem to us moderns. Greek mathematicians must have tacitly learned the tip: when you treat a proportion between two ratios whose terms are not homogeneous, try to make them homogeneous. And they also knew how to use *Elements* II.5 as a tool to determine the maximum of a rectangle contained by two segments of a line. Here again, it is not enough to know a proposition, but one has to know how to use it as a tool.

2.4 Determination of the tangent of more complex curves

The power of analysis as a tool for investigation in geometry is best appreciated in more complicated problems. In the appendix, we have also reconstructed analyses for the determination of the tangent to a hyperbola/ellipse (*Conics* I.34), and to a spiral (Archimedes *On Spirals*, 18).

The conclusions (the properties of the tangents) are simple and elegant and it seems possible to guess them beforehand—by analogy with the tangent of circle for the ellipse/hyperbola, and by kinematic consideration for the spiral. However, the property of the tangent appears in the premise of each proposition, from which the demonstration begins, and, as is usually the case in Greek mathematical works, the author does not explain how that property has been found. The demonstrations are of course correct, but very difficult to follow, for they depend on unexpected auxiliary constructions which look

like *deus ex machina*. So the readers are left with the question “how could the author find this demonstration?”

Our reconstruction of the analysis starts from the simple assumption that the tangent is drawn and it falls always outside the curve except the point of contact, and from this the property of the tangent is found. So both the conclusion and the demonstration of these complicated propositions concerning the tangents can be found by analysis. Though we have no documents discussing how these properties were discovered, we can at least say that we do not have to suppose any other method besides analysis.

3 Limitation of geometric analysis

Now we turn to the defects or limitations of geometric analysis, pointing out two of them. First, as it treats the geometrical objects as they are, in the configuration where they appear, the number of possible cases of a proposition tends to be multiplied. It is not rare that in a proposition dozens of possible cases are hidden. If one tries to make an exhaustive argument, it tends to be intolerably long, and this, indeed, happens in one of Apollonius’ work, which we will examine.

The second point is that the problems which could be invented and investigated were limited. This limitation also comes from the geometry itself, but there is another factor. The only mathematical language for handling geometrical magnitudes was ratio and proportion. This entails difficulties in making and resolving problems which correspond to our equations of third and higher degrees.

3.1 Apollonius’ *Sectio rationis*

Pappus reports some works of Apollonius concerning analysis. Most of them are lost, but one work, *Sectio rationis* (*Cutting off of a ratio*), is extant in Arabic, and has long been known to us through Edmund Halley’s Latin translation [Apollonius 1706]. Recently, the Arabic text with a French translation was published [Apollonius 2010]. English translation [Apollonius 1987] is also available.

This work treats only one problem, divided into many cases. The problem is: given two infinite straight lines on a plane, and a point on each of them, and given a ratio, and besides, given a point not lying on the given lines; to draw a straight line from the given point outside the given lines, so that the segments which are cut off on each line from the given point on it, have the given ratio between them.

Apollonius divides the problem according to the arrangement of given lines and given points, and call each possible arrangement a “locus”, then, each

locus is divided into cases, according to the possible ways of cutting given lines. In each case, Apollonius gives an analysis which may be accompanied by a diorism (examination of the condition for the solution to exist: see the final part of the note 1 above), then the synthesis follows. We are going to examine only the analysis part of one of the cases.

We take the first case of the sixth locus. In this locus, the given straight lines AB and CD meet at point E (Fig. 5), and the given point on AB is the intersection point E itself, while the given point on CD is G. And H is the given point from which the cutting line JL is drawn. And HK, parallel line to AB through H is supposed to meet CD at point K, which is in the prolongation of EG. Then point K is given, so that HK is also given in position and magnitude.⁸

Figure 5: *Sectio rationis*, Book 1, locus 6, case 1. The segments EJ and GL, which will be shown to have the given ratio to each other, are drawn in thick lines.

In the first case of this sixth locus (see Fig. 5), the segment EJ on AB is cut off on the side of the point H, and the segment GL on CD is cut off in the prolongation of EK. Let the line JL be drawn, so that the ratio of the segments EJ:GL is given, i.e.,

$$EJ : GL = \text{the given ratio.} \quad (8)$$

To be able to investigate this problem, there are two things to be done. Firstly, one has to express the given ratio as a ratio of two given lines in the diagram. Although we can express a ratio as an abstract value, say r , this was not the case in Greek mathematics, and a given ratio had to be expressed as a ratio of two lines in the diagram. Secondly, since there are two undetermined (i.e., not given) points J and L on the two different straight lines AB and CD, one has to “move” one of the points to the other line, so that the required condition can be argued on one straight line. Apollonius takes a point M on

8. Because the point H is given and HK is drawn parallel to the given line AB, the line HK is given in position. And the line CD is also given in position, so that the point K is also given. Then, because points H and K are given, HK is given in position and in magnitude.

AC, which satisfies the condition

$$HK : GM = \text{the given ratio.} \quad (9)$$

Since HK is given in magnitude, and the ratio HK : GM is given, GM is also given in magnitude. And the point G is given, so GM is given in position and in magnitude, so that M is given. Thus, the given ratio is expressed as a ratio of two given line segments HK and GM. This relation is also used to eliminate the point J, one of the undetermined points.

Then, since the ratio $\underline{JE}:\underline{GL}$ is the given ratio (hereafter we underline the undetermined points J and L),

$$\underline{JE} : \underline{GL} = HK : GM. \quad (10)$$

Alternately (*Elements*, V.16),

$$\underline{JE} : HK = \underline{GL} : GM. \quad (11)$$

Because JE is parallel to HK,

$$\underline{JE} : HK = \underline{EL} : \underline{LK} \quad (12)$$

From (11), (12), we have

$$\underline{EL} : \underline{LK} = \underline{GL} : GM. \quad (13)$$

Thus the point J has been eliminated, and all the terms of the proportion (13) are found on the line CD. This is a significant advance toward the solution. In the proportion (13), E, G, M are given points, and L is the only point which is not given. But this point L appears in three of the four terms. Apollonius proceeds by reducing the number of the terms in which L appears. By separation of ratio (*Elements* V.17; to take the difference of the antecedent and consequent instead of the antecedent itself), we get

$$EK : \underline{LK} = \underline{LM} : GM. \quad (14)$$

Now we have the undetermined point L in two terms only. Transforming the proportion into the equality of rectangles, we will have the point L on only one side of an equality:

$$r(\underline{KL}, \underline{LM}) = r(EK, GM). \quad (15)$$

The right side $r(EK, GM)$ is given, for EK and GM are given. Therefore, the left side $r(KL, LM)$ is also given in magnitude. And K and M are given points. This is a well-known problem of application of areas with square excess.⁹

9. Our examination suggests the reason why application of areas with excess or defect was one of the fundamental techniques in Greek geometry. If a problem is

3.2 Multiplication of cases

We skip the diorism (examination of the existence and the number of the solution) and synthesis (demonstration) which follow the analysis, and we turn to the analyses of other cases.

In case 2, the point L is between G and K, so that the point J is on the other side of the line CD (Fig. 6). In the third case, the point L falls between E and G (Fig. 7), and consequently the point J is on the other side of the line CD. Then, in the fourth and last case, L is supposed to be on the other side of the line AB (Fig. 8),

Figure 6: *Sectio rationis*, Book 1, locus 6, case 2.

reduced to one proportion in which there is only one undetermined point X, and all the terms of the proportion are line segments on one straight line—this is the case of the present problem we examine—then the problem is solved by transforming the ratio so that not more than two terms contain the undetermined point. If the two terms containing the undetermined point X are the antecedent and the consequent of a ratio, i.e., the condition is reduced to the form

$$AX : XB = CD : EF,$$

then the problem is reduced to cutting the given line AB into given ratio CD : EF at point X. This is a division of a line into a given ratio (internally or externally).

If, on the other hand, the two terms containing the undetermined point X are both inner terms or both outer terms of a proportion, then, transforming the proportion to an equality of rectangles, the problem has the form

$$r(AX, XB) = r(CD, EF).$$

Since the rectangle on the right side of the equality is given, the problem is reduced to that of application of areas with defect (if X is between A and B), or with excess (if X is in the prolongation of AB).

Therefore, the solution of a problem expressed by a proportion leads either to the division of a line in a given ratio, or to the application of areas with excess or defect.

This consideration suggests that we don't have to assume the concept of quadratic equation to explain the origin of the application of areas with excess or defect in Greek geometry.

the hyperbola, ellipse and circle, Eutocius says that there are 12 cases for hyperbola, and 42 cases for ellipse and circle—we would regard a circle as a special ellipse, and would not count its cases separately—so that the total number of the cases is $12+42+42=96$. It seems as if Eutocius were quite content to be able to count the number of cases accurately. For us, this is not very productive work for a mathematician.

4 Concluding remarks: slash-and-burn mathematics

Another particularity with Greek geometry was that the problems that could be conceived and posed were limited. Since ratios and proportions were the main means to express the relations between geometrical objects, it was not easy to formulate a problem beyond those corresponding to quadratic equations. A proportion such as “ a is to b so c is to d ” is reduced, in algebraic terms, to the equality $ad = bc$, so if all the terms a, b, c, d are lines, the relation is equivalent to a quadratic equation. If, for example, a, b are areas while c, d are lines, the relation can be equivalent to a cubic equation. This is indeed the case of the problem of dividing a sphere into a given ratio, treated by Archimedes.¹¹ However, we find but few examples of such problems (“of higher degree” in modern terms) in Greek geometry. They were less often posed, and rarely solved.

Many of the problems in Greek geometry were those which we moderns can reduce to quadratic equations, and these were the problems that Apollonius treated in exhaustive manner in his lost works on analysis reported by Pappus, of which the only extant one is the *Sectio rationis*, which we have examined.

Thus, “why did the analysis of geometric problems decline after Archimedes and Apollonius?” seems to be a bad question. The limitation was set by the mathematical language and available techniques. And the limitation existed not only in the techniques for solution, but also in the range of the problems the mathematicians could think of.

When a theory or a technique for solution was invented in Greek mathematics (we can imagine the conic section, or the exhaustion method, for example), it was exploited as far as possible, and the problems which could be posed and resolved within that theory and technique were investigated. When most of the possible problems were resolved (or left unresolved), there was little room left for further development. In this view, the history of Greek mathematics was a series of invention/ exploitation/ consumption of theories, techniques, and problems. It looks like slash-and-burn agriculture.

11. *Sphere and Cylinder* II.4, and Eutocius’ commentary to this proposition. For detailed examination, see [Netz 1999a, 2004].

To conclude, we repeat that analysis is much more powerful than we usually think, if one knows several “tips” in geometry to enhance its potential power. The limitation to geometrical analysis comes largely from the simple fact that it was geometry. It always treated geometrical objects where they are and as they are. There was no notion of abstract quantity. However, Greek mathematicians probably were not even aware of this limitation. Though they may have lamented the lack of nice problems when they had exploited and consumed them, but did not lament the lack of powerful methods or theory. Analytical geometry was lacking for Archimedes and Apollonius no more than a radio telescope for Galileo.

5 Appendix: Reconstructed analyses of problems

5.1 Tangent to the hyperbola and the ellipse in Apollonius’ *Conics*

Apollonius establishes the property of the tangent to the hyperbola and the ellipse in *Conics* I.34. We propose an analysis reconstructed from Apollonius’ demonstration. Our point is that an analysis could certainly find the property of the tangent, without knowing it beforehand through some other considerations such as by analogy with the circle.

Let AB be the diameter of a hyperbola/ellipse (Fig. 9), E the intersection of the tangent at Γ and the diameter AB, and let $\Gamma\Delta$ be the ordinate drawn from Γ . Apollonius begins with the assertion that

$$A\Delta : \Delta B = AE : EB, \quad (1)$$

without explaining how he has decided on this starting point. Then, from this assumption, he proves that ΓE is, indeed, tangent to the section. As in the case of parabola (§2.3 of the present article), the proof assumes that a point Z on the straight line $E\Gamma$ falls inside the section (it is depicted as such in the diagram, which we have changed), and obtains a contradiction. This is, practically, the same as proving that any point Z on the straight line $E\Gamma$ (except Γ itself) is outside the section.

In the proof, from both extremities of diameter AB, straight lines $AO\Xi$ and BMK are drawn parallel to $E\Gamma$, but the intention of this construction is not explained. Then, from (1), Apollonius derives (after several complicated steps)

$$AN = N\Xi, \quad (2)$$

then, from this equality, he deduces the inequality

$$ZH > \Theta H. \quad (3)$$

Figure 9: *Conics* I.34 (hyperbola/ellipse)

This shows that the line EFZ is the tangent at the point Γ (in the context of Apollonius' proof by contradiction, this is the desired contradiction).

The readers of the *Conics* are at a loss partly because they cannot understand how Apollonius has found the relation (1), but mainly because of the desperately long and complicated argument. As for (1), a possible conjecture is that it was obtained by analogy with the tangent of a circle, but throughout the *Conics*, an ellipse is not treated as a prolonged or compressed circle. Even if one accepted this conjecture, how the demonstration has been found would remain a riddle.

We show how an analysis can find (1), using only the property of tangent, i.e., the inequality (3). What we propose is a reconstruction of an analysis, which is practically Apollonius' demonstration of I.34 put in reverse order. We have added our comments to some key inferences, explaining how they are "natural" and can be found without being aided by any mysterious revelations. We hope to show, again, that analysis is quite powerful if you know some "tips", especially those concerning the use of theory of proportion in geometry. In the following, the labels such as (2.11) are numbered in the order of the synthetic proof in the text of the *Conics* I.34. So the reader will encounter them roughly in reverse order in this reconstruction.

Let us start, following the method of analysis, by assuming that the tangent ΓE has been drawn (although the position of E is not yet known). Then, any point Z on the tangent (except Γ) is outside the conic section, i.e.,

$$ZH > \Theta H. \quad (3)$$

This can be transformed into:

$$q(\Gamma\Delta) : q(\Theta H) > q(\Gamma\Delta) : q(ZH). \quad (2.13)$$

This transformation is motivated by the following proportions which come from the property of hyperbola/ellipse and similar triangles:

$$r(A\Delta, \Delta B) : r(AH, HB) = q(\Gamma\Delta) : q(\Theta H), \quad (2.11)$$

$$q(\Delta E) : q(EH) = q(\Gamma\Delta) : q(ZH). \quad (2.12)$$

Replacing the ratios on each side of (2.13) with the right sides of (2.11) and (2.12), we have

$$r(B\Delta, \Delta A) : r(AH, HB) > q(\Delta E) : q(EH). \quad (2.10)$$

In this proportion, there are two rectangles and two squares, and it is easier to treat them if the inner terms are exchanged so that we have one ratio formed by lines ending at Δ , and another ratio formed by lines ending at H . We have thus

$$r(B\Delta, \Delta A) : q(\Delta E) > r(BH, HA) : q(HE). \quad (2.9)$$

We have come to a deadlock. The difficulty lies in the fact that the consequents of two ratios, $q(\Delta E)$ and $q(HE)$, have different endpoints (Δ and H). Let us transform the proportion so that the both ratio have the same consequents. Making use of the proportion produced by parallel lines, we can change both consequents to $q(\Gamma E)$. This explains why Apollonius draws lines $A\Xi$ and BK .

The details of the argument are as follows. First, let us consider the left side of (2.9). To change ΔE into ΓE , the lines $A\Delta$ and ΔB should also be changed proportionally. To this purpose, the lines AN and BK should be drawn from A and B respectively, parallel to the tangent ΓE . Thus we have

$$r(KB, AN) : q(\Gamma E) = r(B\Delta, \Delta A) : q(\Delta E). \quad (2.7)$$

To obtain a similar transformation of the right side of the proportion, we only have to join ΓH and prolong it to the point M . Then,

$$r(MB, AO) : q(\Gamma E) = r(BH, HA) : q(HE). \quad (2.8)$$

From (2.9), (2.7), and (2.8) we get

$$r(KB, AN) : q(\Gamma E) > r(MB, AO) : q(\Gamma E). \quad (2.6)$$

That is,

$$r(KB, AN) > r(MB, AO). \quad (2.5)$$

This inequality of areas can be better handled if it is transformed to an inequality of ratios, whose antecedent and consequent have common extremities on the left side, A on the right:

$$KB : BM > OA : AN. \quad (2.4)$$

The ratio on the left side can be carried onto the line AO, since

$$N\Xi : \Xi O = KB : BM. \quad (2.3)$$

From (2.4) and (2.3), we have

$$N\Xi : \Xi O > OA : AN. \quad (2.2)$$

In this relation, we have only four points N, Ξ , O, A, all situated on the straight line AO. Of these points N, Ξ and A are given (fixed) in the sense that they do not depend on the position of the point Z arbitrarily taken on the tangent ΓE , and O is the only non-fixed point. So it is natural to bring the lines having the point O to one side of the relation:¹²

$$r(AN : N\Xi) > r(AO, O\Xi). \quad (2.1)$$

Both terms are rectangles, contained by two segments of line A Ξ divided at a point (O or N). Such a rectangle is the greatest when divided at the midpoint of A Ξ (*Elements* II.5). (We have seen this already in our reconstructed analysis of the tangents to the parabola in §2.3.)

Thus, we see that the inequality (2.1) holds regardless of the position O (which depends on the position of Z), if and only if N is the midpoint of A Ξ . That is,

$$AN = N\Xi. \quad (2)$$

Thus we have arrived at (2) starting from (3).

Now we only have to transform the relation (2) so that the position of the point E is determined. This is relatively easy. Considering the ratio of BK to each of AN and N Ξ , from (2), we have

$$BK : AN = BK : N\Xi. \quad (1.3)$$

Because of similar triangles,

$$BK : AN = B\Delta : \Delta A, \quad (1.2)$$

$$BK : N\Xi = B\Gamma : \Gamma\Xi = BE : EA. \quad (1.1)$$

From (1.3), (1.2), and (1.1), we have

$$B\Delta : \Delta A = BE : EA. \quad (1)$$

This is the relation which determines the point E. So we have shown that the tangent of the hyperbola/ellipse can be found by using analysis alone.

12. This manipulation and the transformation from (2.4) to (2.2) are quite similar to that used by Apollonius in *Sectio rationis* which we have examined in §3.1.

5.2 Archimedes' determination of the tangent to spiral

5.2.1 Proposition 18 (first half)

Archimedes, in *On Spirals*, determines the tangent of the spiral. We examine proposition 18 which determines the tangent at the point Θ (Figs. 10, 11), the endpoint of the first rotation of the spiral.¹³ In Figs. 10, 11, the line $A\Theta$ is the start of the rotation, and the point on the spiral reaches Θ after one whole rotation. The straight line AZ is drawn from A at right angles to ΘA .

In this proposition, Archimedes establishes that if ΘZ is tangent at Θ , AZ is equal to the circumference of the first circle, i.e., the circle having the center A and passing through Θ .¹⁴ Hereafter, we denote this circumference by c , and the radius of this circle by r ($= A\Theta$). The proof is divided into two parts.

1. The first part treats the “forward” case, where Archimedes demonstrates that if

$$AZ > c, \tag{1}$$

then the line AZ falls inside the spiral, somewhere in the “forward” part, i.e., after the spiral has completed the first rotation at Θ .

2. The second part treats the “backward” case which is much more complicated. If

$$AZ < c, \tag{2}$$

then the line AZ falls inside the spiral, before it completes the first rotation.

How Archimedes found the equality of the subtangent AZ to the circumference c of the first circle, has been a persistent question. Conjectures of a hidden process aided by kinematic approach or approximating consideration (i.e., with the concept of limit) have been proposed.¹⁵ Here we will argue that

13. We use the critical edition [Archimedes 1910-1915], accepting the proposal in [Archimedes 2017, 61], where Netz rejects an emendation of Commandino followed by Heiberg, restoring the manuscript reading. The diagrams in the present article are not reproductions of the manuscript diagrams, for we have drawn those which we believe serve best for understanding the complicated mathematical arguments.

14. Usually a proposition concerning the tangent to a curve is stated in the manner “such and such a line is the tangent to the curve”. Archimedes states its converse. He proves that ΘZ (with $AZ = c$) is the only possible candidate of the tangent at Θ , but he does not really prove that this line is the tangent. This seems to come from the technical difficulty of the spiral which is a very particular curve.

15. Dijksterhuis proposes an approximating process [Dijksterhuis 1987, 271ff.]. See the next note.

Figure 10: Prop. 18 first half (the Figures in its lemma prop. 7 included)

Figure 11: Prop. 18 “forward” case: enlarged

one can find the tangent and provide its proof by means of analysis alone, without any other alleged heuristic methods.¹⁶

Now, it is intuitively obvious that if AZ is too long, ΘZ (candidate for the tangent at Θ) must fall inside the spiral in the “forward” part of the point Θ , and if AZ is too short, ΘZ must fall inside in the “backward” part.

First, let us assume that AZ is too long, and that a point N in the “forward” part of the point Θ falls inside the spiral. That is, contrary to the diagram, we assume that (Fig. 11)

$$NP < XP, \quad (3)$$

and we look for the minimum length of AZ for a point N satisfying (3) to exist.

The difficulty is that there is no direct way to compare NP and XP . As for the point X , the only available relation is the following, which comes from the fact that X is a point on the spiral:¹⁷

$$XP : \widehat{\Theta P} = r : c. \quad (4)$$

16. As for the analyses proposed by previous scholars, see [Heath 1921, 2:556–561], [Dijksterhuis 1987, 268–274] and [Knorr 1986, 163–166]. Knorr, referring to his predecessors [Knorr 1986, 200, note 51], states that his “own version is not far different from Dijksterhuis’, although less closely wedded to modern differential considerations”. In our reconstructed analysis, we avoid any concept of convergence or limiting value.

17. This relation is derived from

$$XA : r = \widehat{\Theta P} + c : c,$$

which in turn results from the definition of the spiral, through proposition 15.

So we have to find a relation between NP and $\widehat{\Theta P}$, but we have no method to treat the ratio of NP : $\widehat{\Theta P}$.

Instead, we may consider the ratio NP : ΘP , to obtain a sufficient condition for (3) to be true. The reason for this is as follows. If a point N is found such that

$$NP : \Theta P < r : c (= XP : \widehat{\Theta P}), \quad (5)$$

then, since

$$NP : \widehat{\Theta P} < NP : \Theta P, \quad (6)$$

we have

$$NP : \widehat{\Theta P} < XP : \widehat{\Theta P}. \quad (7)$$

This means that NP is smaller than XP, i.e., (3) is true.

So let us look for the condition for AZ which makes (5) true. To compare both sides of (5), it is convenient to make the antecedent of the left side equal to the antecedent of the right side r .

This is easy. In Figure 10, draw the tangent $\Theta\Phi$ of the circle (parallel to AZ) and the line $A\Phi$ parallel to $Z\Theta$ through A, and prolong ΘP to $A\Phi$, and let Y the point of intersection.¹⁸ Then, since

$$NP : P\Theta = AP : PY, \quad (8)$$

and AP is the radius r of the first circle,

$$NP : P\Theta = r : PY. \quad (9)$$

So (5) is true if

$$PY > c. \quad (10)$$

And PY is smaller than $\Theta\Phi$ ($= AZ$), but as Y approaches Φ , the difference becomes as small as one wishes. Therefore, if AZ is greater than c , (10) is true for a certain straight line AN.

Archimedes expresses this fact in a different manner. For any point Λ on AZ such that

$$AZ > A\Lambda > c, \quad (11)$$

he affirms that it is possible to find a line ΘPY such that

$$PY = A\Lambda. \quad (12)$$

18. The labels Φ and Y do not exist in the Archimedean text of proposition 18, but the points corresponding to them are found in proposition 7, which serves as a lemma. See also the following note.

This is a *neusis* construction: find PY equal to AA verging toward Θ . The line ΘY being taken in this manner¹⁹ the point P is determined as the intersection of ΘY and the circle, then the line APN satisfies (5), so that the point N on the straight line $Z\Theta$ falls inside the spiral in the “forward” part of Θ (against our figure).

Archimedes’ demonstration in proposition 18 (and in proposition 7) for the “forward” case of the tangent to the spiral, can thus be found by way of the analysis we have reconstructed above, without any assumed kinematic considerations.

5.2.2 Second half of the proposition 18: backward part

The first half of proposition 18 we have just examined establishes that if AZ is greater than c , the line ΘZ is not the tangent of the spiral at the point Θ , for it falls inside the spiral somewhere after Θ (in the “forward” part of the spiral).

Now, Archimedes demonstrates that if point Z is too near to A for line $Z\Theta$ to be tangent to the curve at Θ , namely

$$AZ < c, \tag{13}$$

(that is, the point Z is too near to the point A), then the line ΘZ falls inside the spiral somewhere in the “backward” part of the curve, before point Θ . This is proved by showing that a line ANII can be drawn from the center A so that AN is smaller than AX (Figs. 12, 13).

19. Archimedes separates this part of the demonstration as proposition 7 and locates it among preparatory lemmata before the definitions concerning the spiral which appear between propositions 11 and 12. So the labels of the points and his way of stating the proposition is different from what we present here, but the nucleus of the argument is the same: it is possible to make PY (verging towards Θ) equal to any given line smaller than AZ.

The possibility of constructing PY verging towards Θ is intuitively obvious. However, this is a *neusis* (inclination) which tended to be excluded from legitimate methods of construction in the development of Greek mathematics. According to Pappus, Apollonius wrote a work in which certain *neusis* problems were resolved by constructions using straight lines and circles (however not all *neusis* problems can be solved by straight lines and circles).

Modern scholars argue about Archimedes’ intention in admitting *neusis* in *On Spirals* ([Knorr 1978] and comments to proposition 7 in [Archimedes 2017]). Knorr, after discussing at length various application of *neusis* by Archimedes, concludes that his effort with *neuses* gave rise to the search for constructions which might eliminate them [Knorr 1986, 178–187]. This explains both Archimedes’ use of *neuses* without hesitation and Apollonius’ effort to replace them by other techniques [see also Knorr 1986, 346]. We limit ourselves to underline that in a work on spirals as a special curve, limitations to the construction techniques would have had little sense, because a spiral itself cannot be generated by straight lines and circles, or even by conic sections.

introducing point Π , the intersection of AN and the tangent of the first circle at the point Θ . From these considerations, the inequality

$$\Theta\Pi > \widehat{P\Theta} \quad (18)$$

holds, so that if one finds a point N such that

$$NP : \Theta\Pi > r : c, \quad (19)$$

one can prove (14) as we argued in (15) and (16). This is the outline of the proof of the second part (“backward” case) of the proposition 18.

Archimedes, assuming (13), takes a point Λ on the prolongation of the line AZ (Fig. 12) such that

$$AZ < A\Lambda < c \quad (20)$$

and draws the line AN such that

$$NP : \Theta\Pi = r : A\Lambda > r : c. \quad (21)$$

The construction of the line AN satisfying (21) which implies (19), is guaranteed by proposition 8, the most difficult one we find in *On Spirals*.

Now we examine this proposition and show that its proof can also be found through geometric analysis.

5.2.3 Reconstruction of the analysis of proposition 8

In the diagram of proposition 8 (Fig. 14), $K\Lambda$ is parallel to YT , so that KY is equal to $\Gamma\Lambda$.²⁰

Moreover, the labels in proposition 8 do not coincide with those in proposition 18. We show here their correspondence here:

Figs. 12, 13	Θ	N	P	Π	Z	Λ
Fig. 14	Γ	E	B	I	(Y)	(Ψ)

The relation to be realized for NP was (Fig. 13)

$$NP : \Theta\Pi = r : A\Lambda. \quad (21)$$

In the context of proposition 8, this becomes (Fig. 14)

$$EB : \Gamma I = K\Gamma : K\Psi, \quad (22)$$

20. In the diagram of proposition 8 (Fig. 14), we have changed the direction of the whole diagram so that the line $\Xi\Gamma\Lambda$ is horizontal, to make the comparison with the diagram of proposition 18 easier. We have also added the labels Y , Ψ and the segment AY , to explain the relation of this lemma to proposition 18.

Then, from (23),

$$\Gamma\Xi > \Gamma\Lambda. \quad (28)$$

This relation will be used later.

Then, from (24) and (27), the condition to be fulfilled is

$$K\Gamma : \Gamma\Xi = EB : \Gamma I, \quad (29)$$

or

$$I\Gamma : BE = \Xi\Gamma : \Gamma K, \quad (30)$$

which is difficult to handle, because BE is not related to any other line segment in this proportion. So, making use of the radius of the circle ($K\Gamma$ or KB), let us transform the ratios and change BE into KE. Since

$$K\Gamma = KB, \quad (31)$$

we have, from (30),

$$I\Gamma : BE = \Xi\Gamma : KB. \quad (32)$$

Taking the remainder (*Elements* V.19) and replacing KB by $K\Gamma$, we have

$$\Xi I : KE = \Xi\Gamma : K\Gamma. \quad (33)$$

Now, to follow the direction of Archimedes' argument further (in reverse order), one has to draw the circle passing through Ξ , Λ and K —an unexpected idea, a *deus ex machina*. Netz points out that the “ankle” $\Xi\Gamma, \Gamma K$ in the ratio $\Xi\Gamma : K\Gamma$, is “highly suggestive of the property of proportion with the meeting of chords in a circle” [Archimedes 2017, 62]. So one may well introduce a circle passing through Ξ and K . True, an excellent mathematician like Archimedes might have immediately seen the circle $K\Lambda\Xi$ in the proportion (33), but a ratio of an “ankle” like $\Xi\Gamma : K\Gamma$ appears quite often, and it is not always handled by drawing a circle.

Instead of assuming that a genius like Archimedes must have seen a circle in this situation, we try to show that even a mediocre mathematician would have ended up drawing the circle, if the given conditions had been carefully examined. Let us suppose that with (33) we have come to a deadlock, not being able to “invent” the circle $K\Lambda\Xi$. In such a case, it is useful to enumerate the given conditions, especially those which are not yet used. There are two such conditions:

$$K\Lambda \parallel E\Gamma, \quad (26)$$

$$\Gamma\Xi > \Gamma\Lambda. \quad (28)$$

proportion, the antecedent of the ratio on the left side is a rectangle contained by $\Xi I, I\Lambda$, i.e., two continuous segments on the line $\Xi\Lambda$, and moreover, the consequent of the ratio $r(KI, \Gamma\Lambda)$ includes KI , a segment which ends at the point I , the dividing point of the segments of the antecedent. At this point, even a mediocre mathematician would be induced to draw the circle $K\Lambda\Xi$ (Fig. 15), because the left side ratio of (38) can be transformed as follows (we introduce point N , the intersection of the prolongation of KI and the circle $K\Lambda\Xi$):²¹

$$r(\Xi I, I\Lambda) : r(KI, \Gamma\Lambda) = r(NI, IK) : r(KI, \Gamma\Lambda) = NI : \Gamma\Lambda. \quad (39)$$

And the right side ratio of (38) can be transformed to

$$\Xi\Gamma : K\Gamma = M\Gamma : \Gamma\Lambda. \quad (40)$$

Now substituting the left sides of (39) and (40) into (38), we get

$$NI : \Gamma\Lambda = M\Gamma : \Gamma\Lambda. \quad (41)$$

Therefore,

$$NI = M\Gamma. \quad (42)$$

Since the circle $K\Lambda\Xi$ and the line $\Xi\Gamma\Lambda$ are given in position, $M\Gamma$ is given in position and in magnitude. Therefore, if we can construct the straight

21. We know, of course, the chord theorem (*Elements* III.35, 36), but not all of us are familiar enough with it to be able to use it in analysis when it is needed. There are at least two useful tips for its application.

1. For any triangle ABC and a point D on the side BC , if one draws a circle ABC and AD is prolonged to meet the circle at E ,

$$q(AD) : r(BD, CD) = AD : DE.$$

See Apollonius, *Conics* II.52, for an example of this claim in use.

2. In the same figure,

$$AD : BD = CD : DE$$

Both are simple consequences of the chord theorem, but usually we are not conscious of them as possible tools for solution of geometric problems.

Greek mathematicians had these and other tips in their “tool-box” which enhanced the power of analysis. If the role of analysis in Greek mathematics tends to be underestimated by us moderns, it is perhaps because we do not know their tool-box, which is equipped with many useful tools, well enough.

line NI satisfying (42) between the circle and the line $\Xi\Gamma$, verging toward K (this is again a *neusis* construction), we can show, by following the analysis in reverse order, that this line satisfies (24). We can then easily give the proof by reversing the order of our analysis. The construction of the line NI by *neusis* satisfying (42) is indeed the starting point of Archimedes' demonstration of the proposition 8.²²

We have thus shown that the proof of proposition 8 can be found by way of analysis. Our analysis takes a different route from the reverse of Archimedes' proof between (33) and (42), because the reconstructed analysis from Archimedes' proof would require an ingenious leap (invention of the circle $K\Lambda\Xi$ from few clues), and we wanted to show that an analysis accessible even to mediocre minds is also possible. Anyway, the equality (42), which resolves the problem can be found by analysis, and we have also shown that an analysis of a problem is not unique, but is flexible to some extent.

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22. Archimedes first takes the point Ξ on the prolongation of $\Gamma\Lambda$ by the proportion (27), then draws the circle $K\Lambda\Xi$, and constructs the chord KIN satisfying (42) by means of the *neusis* construction, affirming its possibility in a terse phrase:

since $\Xi\Gamma$ is greater than $\Gamma\Lambda$, and $K\Gamma$, $\Xi\Lambda$ are at right angles with each other. (Translation by Netz in [Archimedes 2017, 59]. I have deleted the bracket of the word “angle”.)

Although Archimedes is laconic, we can see that he is right. For example, draw the diameter of the circle $K\Lambda\Xi$ parallel to KM (Fig. 15), and let its extremity on the side of the point M be O. Draw KO and let it meet $\Xi\Lambda$ at Π . Then $O\Pi > M\Gamma$. Therefore there exists a chord NIK nearer to Ξ than KO , of which the segment NI is equal to $M\Gamma$.

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