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Analytical Reasoning and Problem-Solving in Diophantus's *Arithmetica*: Two Different Styles of Reasoning in Greek Mathematics

Jean Christianidis

National and Kapodistrian University of Athens (Greece)
Centre Alexandre Koyré, Paris (France)

Résumé: Depuis quelques décennies, la question d'une compréhension historiquement correcte de la méthode de Diophante a attiré l'attention des chercheurs. « L'algèbre moderne (c'est-à-dire, post-viétienne) », « la géométrie algébrique », « l'arithmétique », « l'analyse et la synthèse » sont parmi les contextes proposés par certains historiens pour interpréter les procédures résolutives de Diophante, tandis que la catégorie de « l'algèbre prémoderne » a été proposée récemment par d'autres dans la même finalité. Le but de cet article est d'argumenter contre l'idée de contextualiser le *modus operandi* de Diophante dans le cadre conceptuel de l'analyse ancienne et d'examiner les quelques exemples, dans les livres conservés des *Arithmétiques*, qui pourraient être considérés comme liés aux pratiques qui appartiennent au domaine analytique.

Abstract: Over the past few decades, the question regarding the proper understanding of Diophantus's method has attracted much scholarly attention. "Modern (i.e., post-Vietan) algebra", "algebraic geometry", "arithmetic", "analysis and synthesis", have been suggested by historians as suitable contexts for describing Diophantus's resolutive procedures, while the category of "premodern algebra" has recently been proposed by other historians to this end. The aim of this paper is to provide arguments against the idea of contextualizing Diophantus's *modus operandi* within the conceptual framework of the ancient analysis and to examine the few instances, in the preserved books of the *Arithmetica*, which might be regarded as linked to practices belonging to the field of analysis.

1 Introduction¹

According to a well-known passage in Proclus's *Commentary on the First Book of Euclid's Elements* [Proclus 1873, 203.1–207.25], the formal division of a mathematical proposition comprises six parts: πρότασις (proposition), ἔκθεσις (setting-out, exposition), διορισμός (definition of goal, problem/problem-specification), κατασκευή (construction), ἀπόδειξις (proof, demonstration), and συμπέρασμα (conclusion).² No matter whether this scheme is suitable for describing the canonical exposition of any geometrical (theorem/problem) or arithmetical (like those of Books VII–IX of the *Elements*) proposition,³ it is certainly not appropriate to describe a problem worked-out by algebra, like those in Diophantus's *Arithmetica*.⁴ For example, Diophantus's solutions do not feature any stage corresponding to κατασκευή, with its accompanying diagram.⁵ Moreover, the crucial stage of ἀπόδειξις, in the sense of a textual unit containing chains of deductive steps, like the proofs we read in the *Elements*, is downgraded in Diophantus to a mere test-proof, that is, a verification, which is very often skipped in the Greek books of the *Arithmetica*. Likewise, the part that is designated by διορισμός almost never appears in the Greek books, and only in the books which are preserved in Arabic translation it appears with remarkable consistency. The word itself does not appear in the Greek books, in which we find to be employed four times the cognate noun προσδιορισμός [Diophantus 1893–1895, I, 36.3; 340.9],⁶ and the verb διορίζεσθαι [424.14;

1. This paper was presented to the conference on “Greek Geometrical Analysis: Problems and Prospects” (Clermont-Ferrand, France, March 4–6, 2020). The author would like to thank the organizers for the invitation to participate in the conference, and the anonymous reviewers for their constructive comments.

2. The beginning of this passage is also found in the pseudo-Heronean *Definitions* [Heron 1912, 120.21–122.19]. Translations of the Greek terms have been proposed, among others, by [Netz 1999], [Vitrac 2005], [Acerbi 2011], and [Sidoli 2018b].

3. Sidoli points out that the Proclean scheme is suitable for the simpler problems of the *Elements*, but it fails to capture all of the possible components of the more complicated problems [Sidoli 2018b, 410]. For example, in Euclid's treatment of Proposition III.1 Sidoli distinguishes the following parts: “enunciation”, “exposition”, “problem-specification”, “problem-construction”, “proof-specification”, “proof-construction”, “demonstration”, etc.

4. Cf. [Netz 1999, 293]. However, some scholars still think that the Euclidean format applies to Diophantus's propositions. See for example [Meskens 2010, 57–58], and, more recently, Rashed & Houzel, who claim that “du point de vue architectonique, l'exposé de Diophante épouse délibérément la forme et l'ordre de l'exposé euclidien” [Rashed & Houzel 2013, 23].

5. Cf. Acerbi's statement “In the Diophantine problems construction is almost absent” [Diofanto 2011, 67].

6. References to the Greek books of Diophantus are given according to Tannery's edition [Diophantus 1893–1895]: Vol. I: *Diophanti quae exstant omnia continens* (1893). Vol. II: *Continens pseudepigrapha, testimonia veterum, Pachymerae paraphrasis, Planudis commentarium, scholia vetera, omnia fere adhuc inedita, cum prolegomenis et indicibus* (1895). References to the first volume of this work are

428.21], but with the totally different meaning of the condition of solubility (determination).⁷ By virtue of the above, the scheme described by Proclus is evidently not suitable to describe the structure of the Diophantine proposition.

Recognizing the unsuitability of the Proclean scheme, Sesiano proposed another one, according to which the presentation of a Diophantine worked-out problem comprises the following parts: *πρότασις* (described as “statement of the problem, in terms of required magnitudes and given ones, if any”); *διορισμός* (described as “limiting condition”); *ἔκθεσις* (described as “numerical setting of the given magnitudes”); *ἀνάλυσις* (not described at all); *σύνθεσις* (not described at all); and *συμπέρασμα* (described as “final statement”) [Sesiano 1982, 49]. Sesiano, therefore, regards analysis and synthesis as components of the Diophantine resolatory procedure.

It is true that a characteristic of the Diophantine procedure is to operate with the *quaesitum*—provided that the latter has been named—and this makes the Diophantine solutions apparently analytical [cf. Acerbi in Diofanto 2011, 49].⁸ Interpretations of the Diophantine solutions through the machinery of the ancient analysis are witnessed in the scholarship. Besides Sesiano, one might mention for example the study of [Netz 2012], while the influential [Klein 1968] still remains the *locus classicus* for the idea of contextualising Diophantus's approach within the framework of the ancient analysis.⁹ Concerns about this viewpoint have been raised by Acerbi in the introduction of his [Diofanto 2011]. Acerbi rightly remarks that there are modalities in the Diophantine approach which do not square with the analytical paradigm [Diofanto 2011, 49], and wonders if Diophantus himself

given by simply writing the page number and the line numbers, separated by a dot (.). There are two editions of the Arabic books: [Sesiano 1982] and [Diophante 1984]. References to these books are given according to the edition of Sesiano. All translations from Diophantus are taken from the forthcoming [Christianidis & Oaks 2021].

7. See “Index graecitatis apud Diophantum” [Diophantus 1893-1895, II, 281], s.v. *προσδιορισμός*, and Tannery's explanation “*conditio, limitatio datorum ita ut problema possibile sit*”. According to Acerbi, the word *προσδιορισμός* with the meaning of a condition of solubility may originate in Diophantus himself [Diofanto 2011, 16].

8. Of course this feature is common to any algebraic solution as well. More specifically a solution in premodern algebra comprises naming the unknown(s) in terms of preassigned names of the powers, and working through the operations called for in the enunciation with the named terms, to set up (ideally) a polynomial equation, which is then simplified and solved; finally, the *quaesitum* is calculated by means of the solution of the equation and the assigned name(s). The above are discussed in a number of recent studies, including [Christianidis 2018a,b], [Christianidis & Oaks 2013], [Oaks 2009, 2010a,b], [Oaks & Alkhateeb 2005, 2007] and in the forthcoming [Christianidis & Oaks 2021]. For the algebraic character of Diophantus's *Arithmetica* see, in addition to the above, [Christianidis 2015], [Christianidis & Megremi 2019], [Christianidis & Oaks 2021], [Christianidis & Skoura 2013], [Sialaros & Christianidis 2016].

9. Cf. however n. 23 below.

would have considered his own solutions as analyses. He points out that “he never speaks of ‘synthesis’, but only of ‘demonstration’, which however is always ‘manifest’” [Diofanto 2011]. Besides, even if one were to “qualify the approach in *Arithmetica* not as ‘analytic’ but as operating by ‘reduction’”, as Acerbi suggests, even then the reductions cannot be regarded as relating to an analytical undertaking, since the reasoning is devoid of any connotation of search for preconditions [Diofanto 2011, 50].

The concerns expressed by Acerbi are legitimate.¹⁰ Be that as it may, the issue deserves further investigation, especially since the previous discussions fail to take into account the premodern algebra as interpretative context for Diophantus’s problem solving. The present paper has therefore a twofold aim: first, to provide additional arguments against the idea that Diophantus proceeds in his solutions by analysis-and-synthesis; second, to discuss the very few propositions of the *Arithmetica* which are referred to as “analysis” or “synthesis”.

2 Some discrepancies between Diophantus’s solutions and analytical reasoning

Starting with the first, it is important to note that linguistic analysis does not support subsuming the Diophantine solutions into the field of ancient analysis. An essential feature of any ancient Greek analytical text, either geometric or metrical,¹¹ is the use of the technical language of “givens” [δοθέν and/or δεδομένον] and the unfolding of mathematical arguments shaped as chains of givens, through which the predicate “given” is transferred from one object to another.¹² In Acerbi’s words, the language of the “givens” is “one of the major features of the proof format of analysis and synthesis” [Diofanto 2011, 119]. Yet, this is not the case in Diophantus’s *Arithmetica*. Although the term “given” appears quite often in it, it is not employed in the core part of the propositions, where the working out of the task set forth in the enunciation takes place, nor are the solutions carried out through any chain of givens.

10. However, Acerbi’s reservations are expressed within an interpretative framework which considers Diophantus’s solutions as solutions which proceed by “reductions”, and not as solutions by premodern algebra. In his words “*Arithmetica* are a gigantic heap of reductive procedures” [Diofanto 2011, 50].

11. The term “metrical analysis” was introduced by [Sidoli 2004] to designate analytical practices concerning numerical mensuration of geometrical objects.

12. For recent studies on the meaning and use of the concept of “given” in Greek analytical practices, see [Acerbi 2011], [Sidoli 2018a]. For the historiographical questions concerning the understanding of Greek geometric and metrical analysis see the bibliography provided in Sidoli’s study. For Marinus’s definition of “given”, see [Sialaros, Matera *et al.* 2019].

Let us be more precise. The technical term “given” which is used in analytical texts should be distinguished from the general usage of the term, as, for example, when we say “to write a given number as a sum of two square numbers” or “given a number, to find in how many ways it can be polygonal”, and so on. Such usages cannot be classified as related to analytical practices [cf. Sidoli 2018a, 354]. But this is precisely the case of Diophantus's primary use of the word “given” in *Arithmetica*.¹³ When this word appears in a proposition it appears in the enunciation (generic and/or instantiated) and in the determination, and not in the core part of the solution, that is, the setting up and the solution of the equation.¹⁴ One might say, by repeating the words of Sidoli [Sidoli 2018a, 358], that, in practicing problem-solving by analysis, given objects are given by the problem-solver, whereas given objects (namely, numbers) in the *Arithmetica* are always given by the problem-setter. Therefore, Diophantus does not use the word “given” with the technical meaning it has in ancient analytical texts, with one exception, Problem 10 of the Greek Book V, to be discussed later.

Moreover, Diophantus's use of the words ἀνάλυσις and σύνθεσις is different from that in the ancient analytical texts. The word ἀνάλυσις is never used in the Greek books of the *Arithmetica* with the technical meaning it has in ancient analysis, while the verb ἀναλύειν is used only with the meaning “to reduce in

13. Another use of the word “given”, as part of the formulaic expression “triangle given in form”, occurs often in the Greek Book VI, which deals with rational right-angled triangles. In this context, if (a, b, c) are the sides of a right-angled triangle (that is, if $c^2 = a^2 + b^2$), the expression “triangle given in form” refers to a triangle whose sides are (ax, bx, cx) , which therefore belongs to the class of right-angles triangles which are similar to (a, b, c) . This is the technical meaning of triangle “given in form” according to Euclid's *Data*, Def. 3 [Euclid 1896, 2.6–8], that is, a triangle whose angles are given, or for which the ratios of the sides are given. On the other hand, the word “given” with the technical meaning it has in analytical reasoning is used by Diophantus in his *On Polygonal Numbers*. See [Acerbi 2011], [Diofanto 2011].

14. An example of a problem in which the word “given” is employed in the generic enunciation and the determination is Problem I.9: “To subtract the same number from two given numbers and make the remainders have to one another a given ratio. It is certainly necessary for the given ratio to be greater than the ratio which the greater of the givens has to the smaller” [26.13–18]. An example in which the word appears in the generic enunciation and the instantiation is Problem 4 of Greek Book V: “Given one number, to find three numbers such that any one of them, as well as the product of any two, if it lacks the given, makes a square. Let the given number be 6 units” [318.4–7]. Apart from Problem V.10 (see below), in the Greek books of the *Arithmetica* we find only seven instances in which the word “given” is employed in a part of the proposition other than the enunciation and the determination. In two cases (Problems III.19 and V.13) the word is employed in metamathematical comments aiming at reminding the reader something already shown: “we learned how to divide a given square into two squares” [184.3–4; 350.3–4]; in four cases (Problems III.20, IV.1, Lemma to IV.34, and V.4) it is employed to refer to a specific value assigned in the setting-out [188.1; 190.11; 282.2–3; 6; 318.10]; and in one case (Problem V.3) the word is used in the enunciation of a proposition from the lost *Porisms* [316.8].

the same part” [ἀναλύειν εἰς μόριον].¹⁵ As for the word σύνθεσις, it is used in the Greek books with the meaning of “addition, sum”,¹⁶ and the corresponding verb συντιθέναι with the meaning “to add”.¹⁷ In the Arabic text the words “analysis” and “synthesis” are used a couple of times in Problems 37, 42a, and 43 of Book IV, which will be discussed in the second part of this paper.

Besides linguistic arguments, there is another point on which the problem-solving of Diophantus stands out compared with the analytical problem-solving, as the latter is practiced in the preserved analytical texts. Taking for granted the traditional quadripartite division of a proposition worked-out by analysis-and-synthesis into “transformation”, “resolution”, “construction”, and “demonstration”,¹⁸ in Diophantus’s solutions we do not observe what is very common in almost every instance of an analyzed proposition, that is, the intermingling of steps belonging to different parts. In propositions treated by analysis, steps belonging to “transformation” are often intermingled with steps belonging to “resolution”, as well as steps belonging to “construction” are intermingled with steps belonging to “demonstration” [Sidoli 2018a, 359]. Such intermingling is not possible in Diophantus’s solutions, in fact, in any solution by premodern algebra. For example, the treatment of the equation—which includes its simplification (application of the “restoration” and “confrontation” steps)¹⁹ and solution—can only start when the stage of its setting-up has been accomplished. In other words, the production of the equation, and the solution of the equation are two tasks that cannot be intermingled with each other. Likewise, the “demonstration” (test-proof) can only occur at the very end of the solution. So, in a solution by premodern algebra the steps are clearly separated from one another.

More differences between premodern algebra and the sub-category of ancient analysis called “metrical analysis” have been identified by Sidoli:

There are a number of fundamental differences between the methods of metrical analysis and those of premodern algebra—found, for example, in Diophantus’ *Arithmetics*. For one thing, Diophantus’ procedures are, at least in principle, purely numerical, and do not rely on any underlying geometric conception.

Next, metrical analysis has no special nomenclature for designating objects to be sought, as is introduced in premodern algebra.

15. See “Index graecitatis apud Diophantum” [Diophantus 1893-1895, II, 262], s.v. ἀναλύειν εἰς μόριον, translated by Tannery as “reducere ad denominationem”.

16. [Diophantus 1893-1895, II, 283], s.v. σύνθεσις, translated by Tannery as “additio, summa”.

17. [Diophantus 1893-1895, II, 283], s.v. συντιθέναι, translated by Tannery as “addere”.

18. See [Acerbi 2011, 138–141], [Berggren & Van Brummelen 2000, 5–16], [Fournarakis & Christianidis 2006, 49–50], [Hankel 1874, 137–150], [Hintikka & Remes 1974, 22–26], [Saito & Sidoli 2010, 583–588].

19. For the meaning of these technical terms in the context of the premodern algebraic practice see [Oaks & Alkhateeb 2007].

In ancient analysis, we deal only with geometric objects and with those same objects when they are *given* in various ways—hence, there is no notational, or conceptual, device that allows us to set up a relation containing both *given* elements and explicitly sought elements. A problem in premodern algebra, however, begins with an instantiation of the stated problem using a certain equation, in which terms to be sought are set into relation with actual numbers that are stated to be *given*.

Finally, although metrical analysis provides an algorithm for computing a definite value, it does so in a different manner than premodern algebra. Diophantus' procedure for setting up the equation with which he will work, allows him to set known and sought values on both sides of his equation, so that he can apply operations mathematically equivalent to the arithmetical operations without regard to the epistemological, or ontological, status of any of the objects in his equation. Of course, in metrical analysis we also find equations, and proportions, with non-*given* objects on both sides of the equation, which are then subjected to arithmetical operations and ratio manipulations, but this is done only as an intermediate step before immediately asserting that some geometrical object is *given* in terms of some computable number. In metrical analysis, it is this series of claims about what is *given* that functions as the primary problem-solving procedure. [Sidoli 2018a, 393–394]

To the above arguments one should add that a feature occurring quite often in problem-solving by analysis, whatever geometric or metrical, is that analysis begins with the initial assumption that the problem is solved in its entirety.²⁰ Transferred in the setting of problem-solving by premodern algebra, this would mean that all sought-after numbers called for in the enunciation must be assumed to be known, that is, they have to be named right from the outset of the solution. Against this background, one would expect that any algebraic solution begins with assigning names to *all numbers*, the finding of which the enunciation of the problem calls for. This happens, of course, if there is only one unknown, and in some cases if there are just a few. But in more complicated problems, where several numbers are required, the assignments are done gradually, as the resolutive procedure progresses. This is the case, for example, of all “derivative” assignments, that is, assignments which are based on other assignments. For example, the very first problem of the *Arithmetica*, whose instantiated enunciation we write today as

$$\begin{cases} x + y = 100 \\ x - y = 40 \end{cases}$$

20. This rule is not general; for example, it does not apply to locus problems.

is solved by assigning the smaller sought-after number (designated above by y) to be “1 Number” (which we will write “1N”), thus making the greater sought-after number (designated by x) to be “1 Number, 40 units” (which we will write “1N 40u”). The first assignment is “direct”, the second is “derivative”. The former is made from the outset, the latter presupposes not only the first but also that the addition of “1N” and “40u”, stipulated by the second condition of the problem, has been performed, and the sum (“1N 40u”) has been found.²¹

The above arguments are enough to dismiss the idea that the analysis-and-synthesis format provides a suitable interpretative framework for Diophantus’s method of solution. But there is also another reason for rejecting such an idea: Diophantus describes clearly his method of solution in the introduction of Book I. And he doesn’t have a single word to say about “analysis” and/or “synthesis”. On the contrary, he speaks about a “way” [ὁδός] for solving arithmetical problems, the stages of which are outlined as follows:

1. Generic enunciation.
2. Determination (if necessary).
3. Instantiation (sometimes accompanied by the instantiated enunciation).
4. Set up of the equation (comprising (a) naming the unknowns and (b) working through the operations stipulated in the problem *on the names*).
5. Treatment of the equation (comprising (a) simplification and (b) solution of the simplified form).
6. Answer to the problem.
7. Test-proof.

That the above list portrays the structure of the Diophantine worked-out problem is shown in a number of recent studies.²² Now, the core part of this structure, stages 4 and 5, is the main theme discussed by Diophantus in the introduction, and there is nothing in his text suggesting that his undertaking was conceived to be part of the field of analysis-and-synthesis.²³

21. For the technical terms of premodern algebra, the notions of “assignment”, “named” and “unnamed” term, “forthcoming equation”, and for the techniques used by Diophantus for assigning names to unnamed terms the reader is referred to the papers mentioned in n. 8 above.

22. See the references given in n. 8 above.

23. The idea of contextualizing Diophantus’s *modus operandi* within the conceptual framework of the ancient analysis was advanced by Jacob Klein in his account of Viète’s “reinterpretation of the Diophantine procedure” [Klein 1968, 161ff.]. Klein’s starting point is the idea that the solutions “in the indeterminate form”, that Diophantus occasionally employs in his solutions, “represent a properly ‘analytic’ procedure” [Klein 1968, 164]. Although Diophantus considers such solutions as “merely auxiliary” tools, says Klein, it is possible that a “solution in the indeterminate form” can be applied for every case, with the proviso, of course, that “the numbers ‘given’ in a problem are also regarded only in their character of being given, and not as just these determinate numbers” [Klein 1968, 134]. This step, says Klein,

As said before there are four instances in the preserved books of the *Arithmetica* which might be regarded as linked to practices belonging to the field of analysis. They are found in Problems V^g.10, and IV^a.37, 42a, and 43.²⁴ In what follows, we examine these problems, paying special attention to the relevant aspects.

3 The argument by “givens” in Problem V^g.10 and its function as reduction

Problem V^g.10 asks to cut unit into two parts, and add to each part a different given number, and make a square. Its instantiated enunciation corresponds to the problem we write in modern algebra as

$$\begin{cases} x + y = 1 \\ x + 2 = \square \\ y + 6 = \square' \end{cases}$$

The resolution runs as follows: The conditions of the problem entail that $9u = \square + \square'$, with $2u < \square < 3u$.²⁵ Diophantus assigns the square designated by $\square$ to be 1 Power (we write this assignment as $\square := 1P$), with $2u < 1P < 3u$. Then, he subtracts the $1P$ from $9u$, to obtain $9u$ lacking $1P$ (which we write,

was taken by Viète, who thus arrived at his conception of the “*logistique speciosa*”, which is the true parallel to the ancient geometrical analysis [Klein 1968, 165]. Whatever problems there might be with the above account, what is important to be noticed is that: (a) Klein does not consider any Diophantine solution as a solution by analysis, but only the very few solutions that fall into the category of solutions “in the indeterminate”; (b) even then, he does not consider analysis and synthesis as stages of the resolutive procedure. On this, Klein leaves no room for doubt. “For every case”, he writes, that is, for every solution cast in the form of a solution “in the indeterminate”, “three stages could always be distinguished in the process of solution: (1) the construction of the equation; (2) the transformation to which it is subjected until it has acquired a canonical form which immediately supplies the ‘indeterminate’ solution; and (3) the numerical exploitation of the last, i.e., the computation of unequivocally determinate numbers which fulfill the conditions set for the problem” [Klein 1968, 164]. The identification of these stages with analysis and synthesis, is due, according to Klein, to Viète; ascribing it to Diophantus we are committing a retrospective reading of history.

24. V^g is the Greek Book V, while IV^a is the Arabic Book IV.

25. The relation $9u = \square + \square'$ (i.e., the sum of the given numbers increased by one unit must be the sum of two squares) constitutes a necessary condition that Diophantus omits to state.

$9u \ell^{26} 1P$). Hence, $9u \ell 1P = \square'$. This half-constructed equation²⁷ is easily solved. We simply have to assign the side of the unnamed square to be an expression of the type “ $3u$ lacking some N s”,²⁸ or, in quasi-modern terms, “ $3u \ell mN$ ”, the multitude (i.e., the coefficient m) being any positive rational number. But $1P$ must lie between $2u$ and $3u$. Diophantus takes two squares, one, the smaller, greater than $2u$, the other, smaller than $3u$, and sets out to establish the $1P$ in the interval between the two squares. $\frac{289}{144}u$ (square of $\frac{17}{12}u$), and $\frac{361}{144}u$ (square of $\frac{19}{12}u$), are such squares. So, if we make the multitude (m) to be such that the solution of the equation which emerges from the forthcoming equation satisfies the double inequality $\frac{17}{12}u < 1N < \frac{19}{12}u$, we shall solve the problem.

Now, the equation which arises from the forthcoming equation, after an assignment for the unnamed square of the type mentioned above, is, in quasi-modern terms, $9u \ell 1P = m^2P \ 9u \ell 6mN$ (corresponding to $9 - t^2 = m^2t^2 + 9 - 6mt$), thus $1N$ is $\frac{6m}{m^2+1}$. Therefore, we are looking for a value of m which satisfies the double inequality $\frac{17}{12} < \frac{6m}{m^2+1} < \frac{19}{12}$. This is a subsidiary problem, which Diophantus solves by positing $m := 1N$. Working through the operations we obtain the double inequality $\frac{17}{12}u < \frac{6N}{1P1u} < \frac{19}{12}u$, from which two inequalities emerge:

$$(1) \quad \frac{17}{12}u < \frac{6N}{1P1u}, \text{ or, } 72N > 17P \ 17u$$

$$(2) \quad \frac{6N}{1P \ 1u} < \frac{19}{12}u, \text{ or, } 72N < 19P \ 17u.$$

Diophantus solves the two inequalities by applying the rules for solving the corresponding trinomial equations. For the first he finds successively: Square $\frac{1}{2}72u \rightarrow 1296u$; multiply $17u$ by $17u \rightarrow 289u$; subtract this from $1296u \rightarrow 1007u$; take the square root of this, it is $\not\propto 31u$; take the half of $72u \rightarrow 36u$; add this to $\sqrt{1007u}$, the result is $\not\propto 67u$; divide this by $17u$, the result is $\not\propto \frac{67}{17}u$. Thus, the $1N$ is not greater than $\frac{67}{17}u$. Similarly, for the second inequality he finds that the $1N$ is not smaller than $\frac{66}{19}u$. He adopts for $1N$ the numerical value $3\frac{1}{2}u$, so this is the value of the unknown number of the subsidiary problem designated by m .

We are now in place to continue the solution of the original problem, by making a suitable assignment for the side of the $\square'$. The complete solution is shown in the table below:

26. We use ℓ as the sign standing for “lacking”, by which we translate the Greek $\lambdaείψει$. The corresponding word which is used in the Arabic books is *illā*.

27. Henceforth the expression “forthcoming equations” will be used to designate such half-constructed equations.

28. Tannery’s text has (the abbreviations being resolved) $\muονάδων \bar{\gamma} \lambdaείψει \acute{\alpha}ριθμοῦ \tauινος$ (3 units lacking some Number), but in two of the most important mss., Martinitensis 4678 and Marcianus gr. 308, we find for the last two words the writings $\acute{\alpha}ριθμούς \tauινας$ and $\acute{\alpha}ριθμῶν \tauινων$, both meaning “some Numbers”, that is, “some multitude of Numbers”.

Assignments	Operations	Equation
$\square := 1P(2u < 1P < 3u)$	Subtract $1P$ from $9u \rightarrow 9u \ell 1P$	$9u \ell 1P = \square'$
$\sqrt{\square'} := 3u \ell 3\frac{1}{2}N$	Square $3u \ell 3\frac{1}{2}N \rightarrow$ $12\frac{1}{4}P \ 9u \ell 21N$	$9u \ell 1P =$ $12\frac{1}{4}P \ 9u \ell 21N$

The equation simplifies to $13\frac{1}{4}P = 21N$, hence $1N = \frac{84}{53}u$. The square designated by $\square$ is $\frac{7056}{2809}u$, so, by subtracting $2u$, we find the sought-after number designated by x to be $\frac{1438}{2809}u$, while the number designated by y is found to be $\frac{1371}{2809}u$.

The solution above involves three problems:

$$P1. \begin{cases} x + y = 1 \\ x + 2 = \square \\ y + 6 = \square' \end{cases}$$

$$P2. \quad 9 = \square + \square', 2 < \square < 3$$

$$P3. \quad \frac{17}{12} < \frac{6m}{m^2+1} < \frac{19}{12}$$

At the first stage, the problem $P1$ is reduced to $P2$. At this stage, Diophantus formulates an argument framed in the language of “givens”. The argument has the format of analysis, and this happens only once throughout the preserved books of the *Arithmetica*. The text is the following:²⁹

Let a unit, AB , be set out, and let it be cut at Γ , and let a dyad, $A\Delta$, be added to $A\Gamma$, and a hexad, BE , to ΓB . Therefore, each of $\Gamma\Delta$, ΓE is a square. And since AB is 1 unit, while $A\Delta$ and BE together is an octad, therefore ΔE as a whole becomes 9 units, and these need to be divided into two squares, the (squares) $\Gamma\Delta$ and ΓE . But since one of the squares is greater than $A\Delta$, that is, of a dyad, but smaller than ΔB , that is, of a triad, I am reduced to dividing a proposed square, as in the present case the 9, into two

29. For the Greek text see [336.17–338.9]. Tannery followed Bachet by adding the diagram suggested by the text, for which, however, there is no manuscript basis:

squares, $\Delta\Gamma$ and ΓE , so that one, $\Gamma\Delta$, is in the interval between the dyad and the triad. Indeed, $\Gamma\Delta$ having been found, and the dyad $A\Delta$ being given, then the remainder $A\Gamma$ is given. But AB is 1 unit; therefore $B\Gamma$, a remainder, is given. Therefore, Γ , at which the unit is cut, is, also, given.

The chain of givens at the end of this argument confirms that once the problem *P2* is solved, the original problem *P1* is also solved.³⁰ So, its aim is not to solve the original problem, but to reduce it. Rather than a complete ἀνάλυσις, therefore, we have here an ἀπαγωγή, in which the analytical procedure stops halfway, reaching another problem to solve [cf. Iwata 2016]. Moreover, the analytical argument does not function as a primary problem-solving procedure. This part of the resolution has nothing to do with algebra. The algebraic part of the solution starts after the reduction, it concerns the reduced problem, and features all key characteristics of a premodern algebraic solution: naming the unknowns, working through the operations stipulated by the enunciation, setting up, simplifying, and solving an equation framed in the language of names, and computing the numerical values of the sought-after numbers.³¹

4 Three instances referred to as “analyses” and “syntheses” from the Arabic Book IV

Problem IV^a.37 The problem asks to find a cubic number such that if we multiply the square that comes from its side by two given numbers, and we add what is gathered from each of them to the cubic number, amounts to a square number. Its instantiated version corresponds to the problem we write today as

$$\begin{cases} x^3 + 5x^2 = \square \\ x^3 + 10x^2 = \square' \end{cases}$$

Diophantus’s solution runs as follows. He assigns the sought-after cube (x^3) to be 1 Cube (we write this assignment, $x^3 := 1C$), therefore the square that comes from its side (x^2) is 1 *Māl* (we write, $x^2 := 1M$).³² Working through the operations he obtains the pair of forthcoming equations:

$$\begin{cases} 1C \ 5M = \square \\ 1C \ 10M = \square' \end{cases}$$

30. Proposition 4 of Euclid’s *Data* is repeatedly implied in the deductive steps.

31. Displaying all the above characteristics Problem V^a.10 is closely aligned with Diophantus’s project, so there is no reason for questioning its authenticity.

32. “*Māl*” is a technical term of Arabic algebra, corresponding to the term Power (δύναμις, the square of the algebraic unknown) of Diophantus.

Next, Diophantus must find a square expression (corresponding to $\square$) to equate to the $1C\ 5M$. He wants the side of the square to be “Things”,³³ that is, a multitude of Things. Likewise he must find a square expression (corresponding to $\square'$) to equate to the $1C\ 10M$. In order to find conditions on the respective “multitudes” he argues as follows:³⁴

If we make a side of the square that is a Cube and five *Māls* be Things, then its square is *Māls*. Then, if we subtract the five *Māls* in common from both sides, it leaves a Cube Equals³⁵ *Māls*. Thus it is clear that the number assigned in this problem to be a Thing is equal to the number of the remaining *Māls*. And also, if we make a side of the square that is a Cube and ten *Māls* be Things, then its square is *Māls*. And if we subtract the ten *Māls* in common from both sides, it leaves a Cube Equals *Māls*. Therefore, the number assigned to be a Thing in this analysis is equal to the number of remaining *Māls*. So the remaining *Māls* in the first Equation should be equal to the remaining *Māls* in the second Equation. But the remaining *Māls* in the first Equation is a square number less five units, and the remaining *Māls* in the second Equation is a square less ten units, and so we should find two square numbers such that if we subtract ten units from the greater and five units from the smaller, (they) are equal.

Diophantus deploys in this passage a mental reasoning which runs as follows: if we assign the side of the $\square$ to be of the form mT ,³⁶ the square is m^2M , hence the equation that arises from the first forthcoming equation is of the type $1C\ 5M = m^2M$ (corresponding to $t^3 + 5t^2 = m^2t^2$), thus $1T$ is of the form $m^2 - 5$ (*). Likewise, if we assign the side of the $\square'$ to be nT , the square is n^2M , hence the equation that arises from the second forthcoming equation is of the type $1C\ 10M = n^2M$ (corresponding to $t^3 + 10t^2 = n^2t^2$), thus $1T$ is of the form $n^2 - 10$ (**). Now, from (*) and (**) we obtain that $m^2 - 5$ equals $n^2 - 10$, or, by adding 10 in both sides, $m^2 + 5$ equals n^2 . Thus, two square numbers must be found whose difference is 5 (the smaller of which must be greater than 5). This is a subsidiary problem coinciding with II.10, so Diophantus gives directly the solution $58\frac{7}{9}u$ and $53\frac{7}{9}u$ for the squares n^2 and m^2 respectively. We assign these numbers to be in terms of a *Māl*, instead of a unit, so the assignments for the two unnamed squares are $\square := 53\frac{7}{9}M$ and $\square' := 58\frac{7}{9}M$. Thus the equations which arise from the two forthcoming

33. “Thing” is a technical term of Arabic algebra, corresponding to the term Number (ἀριθμός, the algebraic unknown) of Diophantus.

34. For the Arabic text see [Sesiano 1982, lines 1115–1127, the emphasis is added].

35. The capitalized “Equals” translates the verb *ʿadala* which is used in the statement of an algebraic equation. Equating outside the statement of an equation, which is expressed by some other word, is written “equal” in lower case. For details see [Oaks 2010a].

36. mT stands for the word “Things”, in plural, that is “a multitude of Things”.

equations are:

$$\begin{cases} 1C \ 5M = 53\frac{7}{9}M \\ 1C \ 10M = 58\frac{7}{9}M. \end{cases}$$

After simplification, both equations become $1C = 48\frac{7}{9}M$, therefore $1T = 48\frac{7}{9}u$, so the sought-after cubic number is $\frac{84,606,519}{729}u$.

In the above solution two problems are involved

$$P1. \begin{cases} x^3 + 5x^2 = \square \\ x^3 + 10x^2 = \square' \end{cases}$$

$$P2. \quad m^2 + 5 = n^2, n^2 > 5$$

the latter (P2) being the 10th Problem of Greek Book II. The reduction, which aims at finding suitable assignments for the unnamed squares $\square$ and $\square'$, is expressed as a mental process, and in this context we find the word “analysis” to be used—unlike the above discussed Problem V^s.10, where the similar reduction from the second to the third problem involved was not described as “analysis”.

Problem IV^a.42a Two numbers are required, a cube and a square, such that both the sum and the difference of a cube of the cube and a square of the square are square numbers. In our notation the problem is written:³⁷

$$\begin{cases} (x^3)^3 + (y^2)^2 = \square \\ (x^3)^3 - (y^2)^2 = \square' \end{cases}$$

Diophantus proposes three solutions for this problem. The first, for which he gives only the instructions, is by the method of double-equality. Thus, the stage of the final calculation of the sought-after numbers is not actually pursued, Diophantus being satisfied with merely indicating it by the phrase “after knowing the Thing, we can synthesize everything in the problem”. The other two solutions are in fact identical, the only difference being that in the latter the pair of forthcoming equations is reduced to another pair with smaller “coefficients” and “powers”. Both solutions are described without actually

37. The enunciation calls for the difference to be a square, but it does not say which of the two is the greater. So there are two cases: (1) a cube of the cube subtracting a square of the square, and (2) a square of the square subtracting a cube of the cube. Only the first case is considered here.

working through the calculations. The phrase marking the transition to the stage of the final calculation of the required numbers is, in the second solution, “Then we return to perform the synthesis of the problem”, and in the third, “Once we know it [i.e., the numerical value of the “Thing”], we return to synthesize the problem according to the way we set it up in the analysis”. Below we present the third solution, by adding all implicit calculations.

We assign the side of the sought-after cube (x^3) to be 2 Things (we write this assignment, $x := 2T$), so the cube is $8C$ and the cube of the cube is $512CCC$.³⁸ Likewise, we assign the side of the sought-after square (y^2) to be 4 $M\bar{a}ls$ (we write, $y := 4M$), so the square is $16MM$ ³⁹ and the square of the square is $256CCM$.⁴⁰ Working through the operations we obtain the pair of forthcoming equations

$$\begin{cases} 512CCC \ 256CCM = \square \\ 512CCC \ \ell \ 256CCM = \square' \end{cases}$$

In order to find suitable assignments for the two unnamed squares Diophantus proceeds in the way we saw in the previous problem. The text runs as follows:

If we wish, we can say: five hundred twelve Cube Cube Cubes and two hundred fifty-six Cube Cube $M\bar{a}ls$ Equal a square, and five hundred twelve Cube Cube Cubes less two hundred fifty-six Cube Cube $M\bar{a}ls$ Equal a square. And for any square divided by a square, the result of the division is a square. Thus we divide the five hundred twelve Cube Cube Cubes and the two hundred fifty-six Cube Cube $M\bar{a}ls$ by a square, which can be a Cube Cube $M\bar{a}l$ or four Cube Cube $M\bar{a}ls$ or nine Cube Cube $M\bar{a}ls$ or sixteen Cube Cube $M\bar{a}ls$ or by whatever we wish among the square numbers after we make any one of them Cube Cube $M\bar{a}ls$. As for Cube Cube $M\bar{a}ls$, the result of its division by Cube Cube $M\bar{a}ls$ is a number, and for Cube Cube Cubes, the result is Things. Let us suppose that we divide them by sixteen Cube Cube $M\bar{a}ls$. Then the result of the division is thirty-two Things and sixteen units. And by the same amount we divide this square, let us divide the other square, which is five hundred twelve Cube Cube Cubes less two hundred fifty-six Cube Cube $M\bar{a}ls$, to get thirty-two Things less sixteen units. Thus thirty-two Things and sixteen units are a square, and thirty-two Things less sixteen units are a square. Let us look for a number that, if we add a given (*mafrūd*) number

38. CCC , abbreviation of “Cube Cube Cube”, corresponding to the 9th power of the algebraic unknown.

39. MM , abbreviation of “ $M\bar{a}l \ M\bar{a}l$ ”, corresponding to the 4th power of the algebraic unknown.

40. CCM , abbreviation of “Cube Cube $M\bar{a}l$ ”, corresponding to the 8th power of the algebraic unknown.

to it, which is sixteen, it gives a square, and if we subtract a given (*mafrūd*) number from it, which is sixteen, it gives a square. Once we have found that number, we divide it by thirty-two, so what results from the division is the Thing. *Once we know it, we return to synthesize the problem according to the way we set it up in the analysis.*⁴¹

After reduction of “powers” and “coefficients” of the forthcoming equations, by dividing all terms by $16CCM$, a new pair of forthcoming equations is obtained:

$$\begin{cases} 32T \ 16u = \square'' \\ 32T \ \ell \ 16u = \square''' \end{cases}$$

Following the same procedure as in Problem 37, Diophantus assigns the $\square''$ to be a square multitude of units, say m^2u , so the $1T$ is found to be $\frac{m^2-16}{32}u$, and, similarly, he assign the $\square'''$ to be a square multitude of units, say n^2u , thus the $1T$ is $\frac{n^2+16}{32}u$. Thus, we are reduced again to a subsidiary problem of the type of II.10, and, the rest of the resolution is the same as before.

Problem IV^a.43 The problem asks to find two numbers, a cube and a square, such that a cube of the cube, if we add a given multiple of a square of the square to it, amounts to a square number, and if we subtract a given multiple of a square of the square from it, leaves a square number. In modern notation its instantiated version is:

$$\begin{cases} (x^3)^3 + 1\frac{1}{4}(y^2)^2 = \square \\ (x^3)^3 - \frac{3}{4}(y^2)^2 = \square' \end{cases}$$

We adopt the assignments $x^3 := 1C$, $y := 2M$, and work through the operations to obtain the pair of forthcoming equations:

$$\begin{cases} 1CCC \ 20CCM = \square \\ 1CCC \ \ell \ 12CCM = \square' \end{cases}$$

In order to find suitable assignments for the unnamed squares Diophantus argues as follows:

41. For the Arabic see [Sesiano 1982, lines 1391–1409, added emphasis].

So if we assign (*fara.dnā*) a side of the square Equated to the Cube Cube Cube and the twenty Cube Cube *Māls* to be *Māl Māls*, then its square is *Māl Māls* by *Māl Māls*, of which one of them is called a Cube Cube *Māl*. If we Equate it with <the Cube> Cube Cube and the twenty Cube Cube *Māls*, then we subtract the twenty <Cube Cube *Māls*> in common, it leaves a Cube Cube Cube Equals Cube Cube *Māls*, its number being equal to a square less twenty, and that is the assigned (*mafrūd*) number of a Thing in this solution. Likewise, if we assign (*fara.dnā*) a side of the square Equated to the Cube Cube Cube less twelve Cube Cube *Māls* to be *Māl Māls*, then its square is Cube Cube *Māls*. And if we add to it the twelve Cube Cube *Māls* lacking from the Cube Cube Cube and we make them added in common to both sides, it amounts to a Cube Cube Cube Equals Cube Cube *Māls*, its number being equal to a square and twelve, and that is (also) the assigned number of a Thing in the problem. So a [great] square less twenty Equals a small square and twelve. And we add the twenty in common to the two sides together, giving a small square and thirty-two (which) Equal a great square. The small square is four units, and when thirty-two is added to it, it amounts to thirty-six, which is the great square. So we make the square Equated to the Cube Cube Cube and twenty Cube Cube *Māls* be thirty-six Cube Cube *Māls*, and the square Equated to the other square be four Cube Cube *Māls*. Both Equations turn out to be, after the restoration and confrontation and division, a Thing Equals sixteen units. *We will now perform the synthesis of the problem in the way we did the analysis.*⁴²

Following the usual procedure Diophantus assigns the side of the $\square$ to be, in quasi-modern terms, mMM , the square is m^2CCM , so the equation $1CCC\ 20CCM = m^2CCM$ (corresponding to $t^9 + 20t^8 = m^2t^8$) arises, which is simplified to $1CCC = (m^2 - 20)CCM$ (i.e., $t^9 = (m^2 - 20)t^8$), and, after division by a CCM the $1T$ is found to be of the form $m^2 - 20$. Likewise, if we assign the side of the $\square'$ to be nMM the square is n^2CCM , and so the equation $1CCC\ \ell\ 12CCM = n^2CCM$ (corresponding to $t^9 - 12t^8 = n^2t^8$) arises, which is simplified to $1CCC = (n^2 + 12)CCM$, and, after division by a CCM the $1T$ is found to be of the form $n^2 + 12$. So, $m^2 - 20 = n^2 + 12$. Thus, we are led to a subsidiary problem, namely to find two square numbers whose difference is 32. This problem was solved in II.10, so Diophantus gives directly the solution $36u$ for m^2 and $4u$ for n^2 . We assign these numbers to be in terms of a Cube Cube *Māl*, so the assignments for the two unnamed squares are $\square := 36CCM$ and $\square' := 4CCM$. Thus the equations which arise from the two forthcoming equations are:

42. For the Arabic text see [Sesiano 1982, lines 1474–1494; emphasis added].

$$\begin{cases} 1CCC \ 20CCM = 36CCM \\ 1CCC \ \ell \ 12CCM = 4CCM. \end{cases}$$

After simplification each equation becomes $1CCC = 16CCM$, and, by division, $1T = 16u$. Therefore, the sought-after cube is $4096u$, and the sought-after square is $262,144u$.

Discussion of the three problems of the Arabic Book IV As we saw the words “analysis” and “synthesis”/ “synthesize”, with a meaning which might be regarded as relating to ancient analysis, appear within *Arithmetica* only in Problems 37, 42a, and 43 of the Arabic Book IV. These problems belong to a larger group consisting of Problems IV^a.34–43, whose enunciations are (in our notation):

$$34. \begin{cases} x^3 + y^2 = \square \\ x^3 - y^2 = \square' \end{cases}$$

$$35. \begin{cases} y^2 + x^3 = \square \\ y^2 - x^3 = \square' \end{cases}$$

$$36. \begin{cases} x^3 + 4x^2 = \square \\ x^3 - 5x^2 = \square' \end{cases}$$

$$37. \begin{cases} x^3 + 5x^2 = \square \\ x^3 + 10x^2 = \square' \end{cases}$$

$$38. \begin{cases} x^3 - 5x^2 = \square \\ x^3 - 10x^2 = \square' \end{cases}$$

$$39. \begin{cases} 3x^2 - x^3 = \square \\ 7x^2 - x^3 = \square' \end{cases}$$

$$40. \begin{cases} (x^2)^2 + y^3 = \square \\ (x^2)^2 - y^3 = \square' \end{cases}$$

$$41. \begin{cases} x^3 + (y^2)^2 = \square \\ x^3 - (y^2)^2 = \square' \end{cases}$$

$$42a. \begin{cases} (x^3)^3 + (y^2)^2 = \square \\ (x^3)^3 - (y^2)^2 = \square' \end{cases}$$

$$42b. \begin{cases} (x^3)^3 + (y^2)^2 = \square \\ (y^2)^2 - (x^3)^3 = \square' \end{cases}$$

$$43. \begin{cases} (x^3)^3 + 1\frac{1}{4}(y^2)^2 = \square \\ (x^3)^3 - \frac{3}{4}(y^2)^2 = \square' \end{cases}$$

The solutions of these problems involve mentally conducted reductions to other problems, namely to Problems II.9 and II.10. The table below shows these reductions, giving: (1) the pair of forthcoming equations which result in the course of the resolutive procedure in each case;⁴³ (2) the assignments adopted; and (3) the problem from Book II from which the assignments are found.

43. The unnamed squares indicated in the table by the symbols $\square$ and $\square'$ do not always coincide with the similar symbols in the list of the enunciations given above.

Pb	Forthcoming equations	Assignments	Subsidiary problem
34 ₁	$1C\ 4M = \square, 1C\ \ell\ 4M = \square'$	Solved by double-equality	
34 ₂	$1C\ 4M = \square, 1C\ \ell\ 4M = \square'$	$\square := 20\frac{1}{4}M, \square' := 12\frac{1}{4}M$	$m^2 - n^2 = 8$ (II.10)
35	$4M1C = \square, 4M\ \ell\ 1C = \square'$	$\square := 7\frac{21}{25}M, \square' := \frac{4}{25}M$	$2^2 + 2^2 = m^2 + n^2$ (II.9)
36	$1C\ 4M = \square, 1C\ \ell\ 5M = \square'$	$\square := 25M, \square' := 16M$	$m^2 - n^2 = 9$ (II.10)
37	$1C\ 5M = \square, 1C\ 10M = \square'$	$\square := 53\frac{7}{9}M, \square' := 58\frac{7}{9}M$	$n^2 - m^2 = 5$ (II.10)
38	$1C\ \ell\ 5M = \square, 1C\ \ell\ 10M = \square'$	$\square := 9M, \square' := 4M$	$n^2 - m^2 = 5$ (II.10)
39	$7M\ \ell\ 1C = \square, 3M\ \ell\ 1C = \square'$	$\square := 6\frac{1}{4}M, \square' := 2\frac{1}{4}M$	$m^2 - n^2 = 4$ (II.10)
40	$16MM\ 64C = \square, 16MM\ \ell\ 64C = \square'$	$\square := 31\frac{9}{25}MM, \square' := \frac{16}{25}MM$	$4^2 + 4^2 = m^2 + n^2$ (II.9)
41	$64C\ 16MM = \square, 64C\ \ell\ 16MM = \square'$	$\square := 36MM, \square' := 4MM$	$m^2 - n^2 = 32$ (II.10)
42a ₁	$512CCC\ 256CCM = \square, 512CCC\ \ell\ 256CCM = \square'$	Solved by double-equality	
42a ₂	$512CCC\ 256CCM = \square, 512CCC\ \ell\ 256CCM = \square'$	Not pursued	$m^2 - n^2 = 512$ (II.10)
42a ₃	$32T\ 16u = \square, 32T\ \ell\ 16u = \square'$	Not pursued	$m^2 - n^2 = 32$ (II.10)
42b	$256CCM\ 512CCC = \square, 256CCM\ \ell\ 512CCC = \square'$	Not pursued	$512 = m^2 + n^2$ (II.9)
43	$1CCC\ 20CCM = \square, 1CCC\ \ell\ 12CCM = \square'$	$\square := 36CCM, \square' := 4CCM$	$m^2 - n^2 = 32$ (II.10)

Two kinds of assignment are used in the solutions of the above problems, which might be characterized as “effective” and “virtual” (or “assignments in the indeterminate”). “Effective” is, for example, the initial assignment in Problem IV^a.37, which is introduced by the sentence “We assign the cube to be one Cube”. “Virtual” is the assignment, in the same problem, which is introduced by the conditional phrase “If we make a side of the square [...] be Things,” that is, “[...] be [a multitude of] Things”, the multitude being unspecified. Thus, we have in this problem six assignments, four of which are “effective” and two “virtual”. The “effective” assignments are, 1) $x^3 := 1C$; 2) $x^2 := 1M$; 3) $\square := 7\frac{1}{3}T$; 4) $\square' := 7\frac{2}{3}T$, while the “virtual” assignments, which we give in quasi-modern notation, are, 5) $\square := mT$; 6) $\square' := nT$. The last two assignments may be considered as “assignments in the indeterminate”, in the same sense that the solutions to the lemmas to Problems IV^a.34–36 are “solutions in the indeterminate” [ἐν τῷ ἀόριστῳ], whose role is to provide assignments to be used in the problems that follow.

Now, according to Klein, “the calculation ending ‘in the indeterminate’, which Diophantus uses only as an auxiliary procedure, must be understood as the true analogue to geometric (‘problematical’) analysis” [Klein 1968, 163]. Under such a perspective, the “virtual” assignments 5) and 6) in Problem IV^a.37 may be considered as belonging to the “transformation” part of an analytical reasoning, which ends with a known problem, namely the problem $n^2 - m^2 = 5$ (II.10). The “resolution” part of this analysis would be framed this way: Once a solution of the problem $n^2 - m^2 = 5$ is given, the multitude (m) of Things in the assignment of the side of $\square$, and the multitude (n) of Things in the assignment of the side of $\square'$ are also given. Of course such “resolution” does not exist in the text. Always under this perspective, the “construction” part of the procedure would be the production of the “effective” assignments 3) and 4) from the “virtual” assignments 5) and 6).

If we adopt this explanation for the word “analysis” in the text of the three problems, then we must admit that the analytical reasoning refers not to the whole solution but only to a task pertaining to a specific stage of it, that of finding assignments for the two unnamed squares. In this sense, analysis would be a heuristic tool to carry out only this task. However, textual evidence does not support this interpretation. For in Problems 42a and 43, just after the finding of the numerical value of the algebraic unknown (the “Thing”), and before the next stage, which is the calculation of the numerical values of the sought-after numbers, there is a sentence which marks the transition from one stage to the next. In Problem 42a, for which the text offers three solutions, the sentence in the first solution is “After knowing the Thing, we can synthesize everything in the problem”; in the second solution is “Then we return to perform the synthesis of the problem”; and in the third solution is “Once we know it [i.e., the numerical value of the “Thing”], we return to synthesize the problem according to the way we set it up in the analysis”. In Problem 43 the sentence is, “We will now perform the synthesis of the problem in the way

we did the analysis”. These phrases seem to imply that the word “analysis” refers to the whole series of assignments, and not only to the assignments for the two unnamed squares. Indeed, only in this manner is the allusion to “the synthesis of the problem” consistent with the calculation of all numbers for which the finding the enunciation of the problem calls.⁴⁴

Let us recapitulate: Problems 37, 42a, and 43 belong to a group of ten problems of the Arabic Book IV of the *Arithmetica*, in which a mental process is employed to reduce the task involved in a specific step of the resolutive procedure into a known problem from Book II. The words “analysis” and “synthesis”/ “synthesize” occur only in these three problems of the group. With the exception of the first occurrence of the word “analysis” in Problem 37, all other occurrences appear in sentences which mark a transition to the final stage of the solution, which is the calculation of the sought-after numbers. The table below shows the wording of the transition in the ten problems of the group:

Pb	Without sentence marking the transition	With sentence marking the transition
34 ₁	Let us then divide all of that by a <i>Māl</i> , to get one Thing Equals five units. Therefore a side of the cube is five units, and the cube is one hundred twenty-five...	
34 ₂	Therefore, the one Thing Equals sixteen units and a fourth of a unit. Since we assigned a side of the cube to be one Thing, a side of the cube is sixteen and a fourth...	
35	[...] it gives three and four fifths and a fifth of a fifth of one Equal one Thing. Since we assigned the cube to be from a side of one Thing, a side of the cube is ninety-six parts of twenty-five parts...	
36	[...] to get one Thing Equals twenty-one. Since we assigned a side of the cube to be one Thing, a side of the cube is twenty-one...	

44. What in the aforesaid problems from the Arabic Book IV is described by the expressions “synthesis of the problem” or “synthesize the problem” pertains to the textual unit which in the Greek books is headed by the clause “To the hypostases” [Ἐπὶ τὰς ὑποστάσεις]. For the meaning of this clause, see [Christianidis 2015].

37	[...] it gives one Thing Equals forty-eight units and seven ninths of a unit. And since we assigned a side of the cube to be one Thing, its side is four hundred thirty-nine ninths...	
38	[...] and the one Thing Equals fourteen units. And since we assigned a side of the cube to be one Thing, its side is fourteen...	
39	Therefore the Thing is three fourths of one. The Cube is twenty-seven eighths of an eighth, and a square of a side of the Cube is thirty-six eighths of an eighth.	
40	Therefore the one Thing is what results from dividing one thousand six hundred by three hundred eighty-four, which is four units and a sixth of a unit. Since we assigned a side of the square to be two Things, a side of the square is eight units and a third of a unit...	
41	So the one Thing is three units and a fifth of a unit. Since we assigned the square to be from a side of two Things, <its side> is six units and two fifths of a unit...	
42a ₁		And once the Thing is found, we can return to the hypostases we established. After knowing the Thing, we can synthesize everything in the problem.
42a ₂		[...] and from that we can find the Thing whose value we are looking for. Then we return to perform the synthesis of the problem.
42a ₃		so what results from the division is the Thing. Once we know it, we return to synthesize the problem according to the way we set it up in the analysis.

42b	Therefore, the one Thing is twelve parts of twenty-five. And since we assigned a side of the cube to be two Things, a side of the cube is twenty-four parts of twenty-five parts of a unit...	
43		We will now perform the synthesis of the problem in the way we did the analysis.

The only occurrence of the word “analysis” in a non-transitional sentence is found in Problem 37. Referring to the second equation of the pair of forthcoming equations:

$$\left\{ \begin{array}{l} 1C \ 5M = \square \\ 1C \ 10M = \square' \end{array} \right.$$

the text has the phrase “the assigned number in this analysis is a Thing equal to the number of remaining *Māls*”. However, a few lines above, referring to the first forthcoming equation, the phrase has “problem” instead of “analysis”: “the number assigned in this problem to be a Thing is equal to the number of the remaining *Māls*”. It seems, therefore, that the use of word “analysis” is due to a scribal mistake. This is corroborated by the fact that all other problems of the group 37–43 which contain similar phrases, have “problem”, as shown in the table below:

Pb	Forthcoming equations	
34 ₂	$1C \ 4M = \square$ $1C \ \ell \ 4M = \square'$	“the number that was assigned in the <i>problem</i> to be one Thing...” “the number that was assigned in the <i>problem</i> to be a Thing...”
35	$4M \ 1C = \square$ $4M \ \ell \ 1C = \square'$	“the number that was assigned in the <i>problem</i> as <a Thing>...” “the number assigned in the <i>problem</i> to be a Thing...”
37	$1C \ 5M = \square$ $1C \ 10M = \square'$	“the number assigned in this <i>problem</i> to be a Thing...” “the assigned number in this <i>analysis</i> is a Thing...”
38	$1C \ \ell \ 5M = \square$ $1C \ \ell \ 10M = \square'$	“the number taken to be Things in the <i>problem</i> ” “the number taken to be Things in the <i>problem</i> ”

As for the transitional sentences in Problems 42a–43 which contain the words “analysis” and “synthesis”/ “synthesize,” they are in all probability

later additions. The extreme case that the problems as a whole are not genuine cannot be excluded, especially if one takes into account that in the *Fahrī* of al-Karājī, which borrows heavily from Diophantus, the last three problems of the Arabic Book IV are omitted [Sesiano 1982, 60].

5 Conclusion

Let us provide a brief summary of what we discussed in this paper. In Section I we highlighted the unsuitability of two schemes which have been proposed for describing the structure of the propositions in *Arithmetica*, the Proclean and the analytical. A weakness that both schemes share in common is that they do not take into account that Diophantus's solutions are solutions by premodern algebra. In Section II we investigated in particular some discrepancies between the analytical style of reasoning and Diophantus's presentation. In Section III we discussed the only occurrence of an analytical argument in the preserved *Arithmetica*, namely the argument by “givens” in Problem 10 of the Greek Book V, and we showed that its function is to reduce the problem into another, and not to solve it. Lastly, in Section IV we investigated the arguments referred to as “analyses” and “syntheses” in three problems from the Arabic Book IV, and we concluded by questioning the genuineness of the relevant passages.

The basic thesis that recurs like a leitmotif throughout this paper is that *Arithmetica* is a work of premodern algebra, and so the solutions it contains not only exhibit the technical vocabulary of Diophantus's algebra but also are structured according to the norms of the premodern algebraic problem solving.

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