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Herman Ruge Jervell

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Constructing Ordinals

Herman Ruge Jervell
University of Oslo

Abstract: We show how to construct ordinals up to the small Veblen ordinal in a constructive way and discuss some of the problems trying to go beyond them.

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1

It must be conceded that Cantor's set theory, and in particular his creation of ordinals, is a grandiose mathematical idea.

Thoralf Skolem [Skolem 1962].

We can obviously construct some ordinals. The ordinal ω is given by the natural numbers. Addition is performed by putting one ordering after the other:

$$A + B : \text{---}\overset{A}{\text{---}}\text{---}\overset{B}{\text{---}}\text{---} \quad A \text{ then } B$$

Multiplication is done by using the lexicographical ordering of pairs:

$$A \times B : \underbrace{\overset{A}{\text{---}}\overset{A}{\text{---}}\overset{A}{\text{---}}\dots\overset{A}{\text{---}}}_B \quad A \text{ copied } B \text{ times}$$

Addition and multiplication is the same for cardinals and ordinals. Cardinal exponentiation is defined by using all functions, while ordinal exponentiation is defined by all functions with finite support. So to define ω^A we consider functions $f : A \rightarrow \omega$ with $fa = 0$ for all but a finite number of the a 's. The graphs of such functions are ordered lexicographically — the values of the larger a 's being more important.

Note that we define the arithmetical operation without invoking that the orderings are well ordered. For the exponentiation we could have defined B^A as long as B has a least element.

2

The operations are defined for orderings and the operations turn out to preserve well orderings.

For the proofs that transfinite induction is permitted we see that for ω it is nothing but ordinary induction. To get addition and multiplication we just use induction over the parts with a slightly more complicated property. A new thing enters with exponentiation. The trick is to replace transfinite induction over ω^A of a property Px with transfinite induction over A on the more abstract property

$$\lambda\alpha.\forall n.P(\omega^\alpha + n)$$

This trick goes back to Gerhard Gentzen (see [Gentzen 1969]) — and was used by him to argue that we get transfinite induction up to any ordinal $< \epsilon_0$ as long as we are allowed to use induction over properties involving more quantifiers. (We use ordinary induction in handling the universal quantifier $\forall n$.)

We know that it is hard to argue constructively for induction up to and including ϵ_0 . Transfinite induction up to ϵ_0 correspond to a Gödel sentence for elementary arithmetic and Gentzen showed that it is equally hard to argue for the transfinite induction up to ϵ_0 as to argue why the Gödel sentence for elementary arithmetic was true.

3

These ordinals Cantor has introduced by a creative process which is very characteristic of his way of thinking. *Thoralf Skolem* (see [Skolem 1962]).

The creative process by Cantor [Cantor 1966] is the following

- We construct ordinals and their ordering
- We start with 0
- Say that we have at some stage constructed some ordinals and the ordering between them. The ordinals constructed so far are well ordered and the set of them gives a new ordinal just larger than what we have already constructed.

Using such a process Cantor could go on and introduce the second, third and higher number classes. But from a constructive point of view it does not make much sense. Here Brouwer was clear. As statement XIII in his thesis of 1907 we have: “Cantor’s second number class does not exist.” [Brouwer 1975].

But we can certainly go to ϵ_0 and beyond in a constructive way. For ϵ_0 we just throw in a new name as a limit to the process of approaching it from below using say exponentiation. But then we can use the arithmetical operations on them and so on. The systematic way of doing this was developed by Oswald Veblen in 1908 [Veblen 1908] — following suggestions from G. H. Hardy [Hardy 1903].

- We construct enumerating functions of ordinals — the functions enumerate closed unbounded sets, or are what we call normal functions.
- We typically start with the normal function ω^α which enumerates the ordinals closed under addition — or some other known normal function.
- To get a new normal function we enumerate fix points of the functions defined already.

We can regard the terms as names for ordinals and with the function enumerating fix points we get a number of new names for ordinals which did not have names already. In this way we get notations for many ordinals. The ordinal ϵ_0 is only the first fix point of the function $\lambda x.\omega^x$ — and is also the first ordinal which we cannot name with terms built up from the arithmetical operations and constants $0, 1, \omega$.

4

In the remainder of this paper we shall work with our own version of the Veblen hierarchy [Jervell 2005]. Instead of working with ordinals as something given and having names for some of them, we work directly with finite trees and give them an ordering which we show in a constructive way to be a well ordering.

Here are some examples of trees and their corresponding ordinals:

$$\begin{array}{cccc}
 \begin{array}{c} \cdot \\ \diagup \quad \diagdown \\ \cdot \end{array} & = \omega & \begin{array}{c} \cdot \\ \diagup \quad \diagdown \\ \cdot \quad | \\ \cdot \end{array} & = \omega^\omega \\
 \begin{array}{c} \cdot \\ \diagup \quad \diagdown \\ \cdot \quad | \\ \cdot \quad | \\ \cdot \end{array} & = \epsilon_0 & \begin{array}{c} \cdot \\ \diagup \quad \diagdown \\ \cdot \quad | \\ \cdot \quad | \\ \cdot \quad | \\ \cdot \end{array} & = \Gamma_0
 \end{array}$$

In a simple way we get some rather complicated ordinals. We start with the finite numbers:

$$\cdot = 0$$

$$\begin{array}{c} \cdot \\ | \\ \cdot \end{array} = 1$$

$$\begin{array}{c} \cdot \\ | \\ | \\ \cdot \end{array} = 2$$

$$\begin{array}{c} \cdot \\ | \\ | \\ | \\ \cdot \end{array} = 3$$

And we have the first infinite numbers:

$$\begin{array}{c} \cdot & & \cdot \\ & \diagdown & / \\ & \cdot \end{array} = \omega$$

$$\begin{array}{c} \cdot & & \cdot \\ & \diagdown & / \\ & \cdot \\ & | \\ & \cdot \end{array} = \omega + 1$$

$$\begin{array}{c} \cdot & & \cdot \\ & \diagdown & / \\ & \cdot \\ & | \\ & | \\ & \cdot \end{array} = \omega + 2$$

And some larger numbers:

$$\begin{array}{c} \cdot \\ | \\ \cdot & & \cdot \\ & \diagdown & / \\ & \cdot \end{array} = \omega \cdot 2$$

$$= \omega \cdot 3$$

Let us define the ordering. We start with finite trees — the root is downmost and the branching is ordered from left to right.

We write $\langle \mathbf{A} \rangle$ for the finite sequence of immediate subtrees of the tree $\mathbf{A}$, and $\langle \cdot \rangle$ is the empty sequence. Equality between trees is the usual equality. Given that we already know the ordering of some trees we let

$\mathbf{A} \leq \langle \mathbf{B} \rangle$: There is an immediate subtree $\mathbf{B}_i$ of $\mathbf{B}$ such that either $\mathbf{A} < \mathbf{B}_i$ or $\mathbf{A} = \mathbf{B}_i$

$\langle \mathbf{A} \rangle < \mathbf{B}$: For all immediate subtrees $\mathbf{A}_j$ of $\mathbf{A}$ we have $\mathbf{A}_j < \mathbf{B}$

$\langle \mathbf{A} \rangle < \langle \mathbf{B} \rangle$: The inverse lexicographical ordering of the immediate subtrees — we first check which sequence have smallest length, and if they have equal length we look at the rightmost immediate subtree where they differ.

We define the ordering of trees by recursion over the immediate subtrees.

$$\boxed{\mathbf{A} < \mathbf{B} \Leftrightarrow \mathbf{A} \leq \langle \mathbf{B} \rangle \vee (\langle \mathbf{A} \rangle < \mathbf{B} \wedge \langle \mathbf{A} \rangle < \langle \mathbf{B} \rangle)}$$

All this is straightforward from a constructive point of view. It is a simple exercise to write computer programs for deciding the ordering.

5

There is more work to calculate the ordinals of the trees above. We distinguish between properties which are given by ordinary induction over the height of the trees and those properties where we need more complicated methods of proofs. For the proofs it may be worthwhile to look at a decision tree for the ordering:

So to decide the ordering between A and B we only need to look at ordering between A or B and subtrees of them. This shows immediately that the ordering is decidable and that equality in the ordering is ordinary equality of trees.

To get transitivity of the ordering we had to consider a number of cases in the decision tree but the proof is again by ordinary induction over the heights of trees.

It is a pleasant property that the equality in the ordering is simply ordinary equality of trees. Each ordinal which corresponds to a tree, corresponds to a unique tree. Below we shall often write down an ordinal instead of the corresponding tree.

The following properties are also proved by induction over heights:

Monotonicity: If we have $A < B$ and tree C where we substitute in A and B at the same place, then $C[A] < C[B]$.

Embedding: If tree A can be embedded into tree B and $A \neq B$, then $A < B$.

6

We now want to do some calculations of ordinals. The empty tree is the smallest tree and corresponds to the ordinal 0.

It is already some effort to show:

$$\begin{matrix} \alpha \\ | \\ \cdot \end{matrix} = \alpha + 1$$

We do this by showing:

$$A < B \Leftrightarrow \begin{matrix} A \\ | \\ \cdot \end{matrix} \leq B$$

or equivalently

$$B < \begin{matrix} A \\ | \\ \cdot \end{matrix} \Leftrightarrow B \leq A$$

by induction over the heights. It is obviously true if one of the trees are empty. The one way of the equivalences follows from:

$$B \leq A < \begin{matrix} A \\ | \\ \cdot \end{matrix}$$

So assume

$$B < \begin{matrix} A \\ | \\ \cdot \end{matrix}$$

Then either B is empty — and we get $B \leq A$ and are done — or we get if the branching in B is unary:

$$B = \begin{matrix} B_0 \\ | \\ \cdot \end{matrix}$$

But then by monotonicity:

$$B_0 < A$$

By induction:

$$B = \begin{matrix} B_0 \\ | \\ \cdot \end{matrix} \leq A$$

If the branching in B is more than unary, then we must use the first clause in the definition of the ordering and we get immediately:

$$B \leq A$$

7

For our constructive ordinals it is essential to see how the ordinals can be built from below. Given a tree $\mathbf{A}$ with immediate subtrees

The immediate subtrees a_i of $\mathbf{A}$ are smaller.

Now assume

- $b_l < a_l$
- $c_i < \mathbf{A}$ for all $i < l$

Then the following tree is less than $\mathbf{A}$

This can be rephrased that for $b_l < a_l$ the function which to $x_0, \dots, x_{\ell-1}$ gives

is closed under $\mathbf{A}$. We also get that for $s < p$ that $\mathbf{A}$ is closed under the function which to $x_0, \dots, x_s$ gives

This is interesting. Given a tree $\mathbf{A}$ we find both smaller trees and operations on trees which are closed under $\mathbf{A}$

Fundamental subtrees of $\mathbf{A}$: The immediate subtrees of $\mathbf{A}$.

Fundamental functions of $\mathbf{A}$: The two types of functions above.

Fundamental set of $\mathbf{A}$: The set of trees generated by the fundamental functions starting with the fundamental subtrees.

Elementary fundamental function of $\mathbf{A}$: We first get unary functions by letting all variables except the rightmost be 0. Then use all such unary functions of the first type. If there are no functions of the first type use the one of the second type with largest branching.

Elementary fundamental set of $\mathbf{A}$: The set of trees generated by the elementary fundamental functions starting with the fundamental subtrees.

We denote the fundamental set of $\mathbf{A}$ with $\mathcal{F}(\mathbf{A})$ and we shall write it as

$$[S, \dots, T | F, \dots, G]$$

where we have displayed the fundamental subtrees $S, \dots, T$ and the fundamental functions $F, \dots, G$. Similarly for the elementary fundamental set $\mathcal{H}(\mathbf{A})$

We have the following:

$$\mathcal{F}(.) = \emptyset$$

$$\mathcal{F}(\dot{\cdot}) = [\cdot |]$$

$$\mathcal{F}(\begin{array}{c} \cdot \quad \cdot \\ \backslash \quad / \\ \cdot \end{array}) = [\cdot | \begin{array}{c} x \\ \cdot \\ \cdot \end{array}]$$

$$\mathcal{F}(\begin{array}{c} \cdot \quad \cdot \\ \backslash \quad / \\ \cdot \quad \cdot \\ \cdot \end{array}) = [\cdot, \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} | \begin{array}{c} x \\ \cdot \\ \cdot \end{array}, \begin{array}{c} y \\ \cdot \quad \cdot \\ \backslash \quad / \\ \cdot \end{array}]$$

$$\mathcal{F}(\begin{array}{c} \cdot \quad \cdot \\ \backslash \quad / \\ \cdot \quad \cdot \\ \cdot \quad \cdot \end{array}) = [\cdot | \begin{array}{c} x \\ \cdot \\ \cdot \end{array}, \begin{array}{c} y \\ \cdot \quad \cdot \\ \backslash \quad / \\ \cdot \end{array} z]$$

$$\mathcal{H}(\begin{array}{c} \cdot \quad \cdot \\ \backslash \quad / \\ \cdot \quad \cdot \end{array}) = [\cdot | \begin{array}{c} \cdot \quad \cdot \\ \backslash \quad / \\ \cdot \end{array} x]$$

Here x, y, z are variables used for describing fundamental functions.

Now we note that the fundamental sets give an approximation of trees from below. In fact we have for any tree $\mathbf{A}$:

$$\mathbf{B} < \mathbf{A} \Leftrightarrow \exists \mathbf{C} \in \mathcal{F}(\mathbf{A}). \mathbf{C} \geq \mathbf{B}$$

We prove this by induction over the height of $\mathbf{B}$. It is trivial for height 0. So assume it proved for smaller heights than the height of $\mathbf{B}$. The direction $\Leftarrow$ is obvious. We assume $\mathbf{B} < \mathbf{A}$ and divide up into cases:

$\mathbf{B} \leq \langle \mathbf{A} \rangle$: But then $\mathbf{B}$ is less than or equal to one of the fundamental subtrees of $\mathbf{A}$.

$\langle \mathbf{B} \rangle < \mathbf{A} \wedge \langle \mathbf{B} \rangle < \langle \mathbf{A} \rangle$: By induction — to each immediate subtree $\mathbf{B}_i$ there is an $\mathbf{C}_i \in \mathcal{F}(\mathbf{A})$ with $\mathbf{C}_i \geq \mathbf{B}_i$. Depending on how we prove $\langle \mathbf{B} \rangle < \langle \mathbf{A} \rangle$ we get a fundamental function which we can apply to some of the $\mathbf{C}_i$'s to get a $\mathbf{C} \in \mathcal{F}(\mathbf{A})$ with $\mathbf{C} \geq \mathbf{B}$

And it is proved. We can also use the elementary fundamental set
For any tree $\mathbf{A}$:

$$\mathbf{B} < \mathbf{A} \Leftrightarrow \exists \mathbf{C} \in \mathcal{H}(\mathbf{A}). \mathbf{C} \geq \mathbf{B}$$

We only need to note that

where $\gamma > \max(\alpha, \beta)$ and that the result of an application of the second type of fundamental function can be embedded into an application of the first type.

With some extra assumption we can conclude that the fundamental set is not only an approximation from below, but that it contains all smaller trees. The extra assumption is that all trees less than or equal to the fundamental subtrees of $\mathbf{A}$ is contained in $\mathcal{F}(\mathbf{A})$. We then have

$$\mathbf{B} < \mathbf{A} \Leftrightarrow \mathbf{B} \in \mathcal{F}(\mathbf{A})$$

The proof follows the lines above. We have induction over the height of $\mathbf{B}$ and get to the cases

$\mathbf{B} \leq \langle \mathbf{A} \rangle$: Then by assumption $\mathbf{B} \in \mathcal{F}(\mathbf{A})$.

$\langle \mathbf{B} \rangle < \mathbf{A} \wedge \langle \mathbf{B} \rangle < \langle \mathbf{A} \rangle$: By induction — for each immediate subtree $\mathbf{B}_i$ we have $\mathbf{B}_i \in \mathcal{F}(\mathbf{A})$. Depending on how we prove $\langle \mathbf{B} \rangle < \langle \mathbf{A} \rangle$ we get a fundamental function which we can apply to the $\mathbf{B}_i$'s to get $\mathbf{B} \in \mathcal{F}(\mathbf{A})$.

We are now getting a clearer picture of the ordering. The trees can be divided into layers — we let $\mathcal{T}_i$ be the trees with at most i -branchings. We then get that $\mathcal{T}_1$ is majorised by

and this tree is the least in $\mathcal{T} - \mathcal{T}_1$. The $\mathcal{T}_2$ is majorised by

and this tree is the least in $\mathcal{T} - \mathcal{T}_2$. The $\mathcal{T}_3$ are majorised by

and this tree is the least in $\mathcal{T} - \mathcal{T}_3$. And so on.

8

So far we have only used ordinary induction over the heights of trees to prove properties, but more is needed to prove that the tree ordering is well founded. We give two constructive proofs.

Using Kruskal's Theorem

Wim Veldman has given a constructive proof of Kruskal's theorem [Veldman 2004]. This can be used. Assume we have an infinite sequence of trees T_i . Then by Kruskal's theorem there must be $m < n$ where T_m can be embedded in T_n , and the infinite sequence cannot be descending.

Using Inductive Definitions

We prove that if $\mathbf{T}_1, \dots, \mathbf{T}_n$ are well founded then so is also all $\mathbf{A}$ with

$$\mathbf{A} < \mathbf{T} = \begin{array}{c} \mathbf{T}_1 \quad \cdots \quad \mathbf{T}_n \\ \diagdown \quad | \quad \diagup \\ \cdot \end{array}$$

and hence $\mathbf{T}$ itself is well founded. This is done by induction over

- the height of $\mathbf{A}$
- the sequence $\mathbf{T}_1, \dots, \mathbf{T}_n$ ordered by the inverse lexicographical ordering of tree ordering

So assume we have $\mathbf{T}_1, \dots, \mathbf{T}_n$ well founded and $\mathbf{A} < \mathbf{T}$. We then have the following cases

- $\mathbf{A} \leq \langle \mathbf{T} \rangle$: Then $\mathbf{A} \leq \mathbf{T}_i$ for some i and is therefore well founded.
- $\langle \mathbf{A} \rangle < \mathbf{T} \wedge \langle \mathbf{A} \rangle < \langle \mathbf{T} \rangle$: Then all immediate subtrees of $\mathbf{A}$ are less than $\mathbf{T}$ and by induction over height of $\mathbf{A}$ the immediate subtrees of $\mathbf{A}$ are well founded. But the immediate subtrees of $\mathbf{A}$ comes before the immediate subtrees of $\mathbf{T}$ in the inverse lexicographical ordering. We conclude that $\mathbf{A}$ is well founded.

We then get that all finite trees are well founded by some giant — and perhaps not so informative — step.

9

The tree characterizing elementary arithmetic is

$$\begin{array}{c} \cdot \quad \cdot \\ \diagdown \quad | \quad \diagup \\ \cdot \end{array} = \epsilon_0$$

and it is the supremum of the trees with unary and binary branching. We can analyze this further. The tree

$$\begin{array}{c} \cdot \quad \alpha \\ \diagdown \quad \diagup \\ \cdot \end{array}$$

is the supremum of trees

$$\left(\begin{array}{c} \cdot \quad \cdot \quad \cdot \\ \diagdown \quad \diagup \quad \cdot \\ \cdot \end{array} \right)^n$$

where $\alpha- < \alpha$ and n is a natural number. Here we use the usual product ordering defined lexicographically. This is enough to conclude

$$\begin{array}{c} \cdot \quad \cdot \quad \cdot \\ \diagdown \quad \diagup \quad \cdot \\ \cdot \end{array} \quad \alpha = \omega^{\omega^\alpha}$$

And we can argue that transfinite induction up to — but not including ϵ_0 — is provable with a trick a la Gentzen.

The fundamental set of a tree corresponds nicely with the Veblen functions. In fact the trees correspond exactly to the fix point free Veblen function of higher arity [Jervell 2005].

10

We argue in a constructive way for ordinals up to the Veblen ordinal. It may be worth noting that it is the Veblen ordinal $\phi_{\Omega^\omega}(0)$ that is interesting. The often cited ordinal Γ_0 does not play any important role — it is given by:

$$\begin{array}{c} \cdot \quad \cdot \quad \cdot \\ \diagdown \quad \diagup \quad \cdot \\ \cdot \end{array} = \Gamma_0$$

11

We can go further with finite labeled trees — and then get as far as Takeuti goes with his ordinal diagrams [Takeuti 1975]. But then we have problems with the proof of well foundedness. We have a proof using minimal bad sequences, and we see no way around that to a constructively acceptable proof.

Let us look closer at the phenomenon without going into the finite labeled trees. We look at proofs of the well foundedness of the tree ordering.

Using Minimal Bad Sequences

Assume we have an infinite descending sequence of trees $\mathbf{T}_i$. There is then a minimal descending sequence $\mathbf{S}_i$. It is minimal in the following sense: For each i : $\mathbf{S}_{i+1}$ is a tree of minimal height among the trees $\mathbf{S}$ where we have an infinite descending sequence starting with $\mathbf{S}_0, \dots, \mathbf{S}_i$, then $\mathbf{S}$, and then continuing with other trees. Then observe

- Because of the minimality we only use the second clause in the definition of the ordering
- For each i : $\langle \mathbf{S}_i \rangle > \langle \mathbf{S}_{i+1} \rangle$
- From some k : For all $i \geq k$ the sequences $\langle \mathbf{S}_i \rangle$ must be of the same length.
- From some ℓ there is an m : We have a descending sequence of the m 'th immediate subtree of $\mathbf{S}_\ell, \mathbf{S}_{\ell+1}, \mathbf{S}_{\ell+2}, \dots$. This contradicts the minimality of $\mathbf{S}_\ell$

The problem enters with the minimal bad sequence. We start with an infinite sequence. Then we choose a minimal $\mathbf{S}_0$, then a minimal $\mathbf{S}_1$, then a minimal $\mathbf{S}_2$ and so on and so on. I do not see any reason why this could be done in a constructive way.

We have seen that there are constructive arguments for the well foundedness of finite trees. The first test case for going beyond this is for the Takeuti ordinal diagrams with labels 0 and 1. They give the ordinals up to the Howard ordinal but a minimal bad sequence argument for their well foundedness.

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