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At the Heart of Analysis: Intuitionism and Philosophy

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For Dirk van Dalen
Victrix causa deis placuit sed victa Catoni

Abstract: One's first impression is that Brouwer's Continuity Theorem of intuitionistic analysis, that every total, real-valued function of a real variable is continuous, stands in straightforward contradiction to a simple theorem of conventional real analysis, that there are discontinuous, real-valued functions. Here we argue that, despite philosophical views to the contrary, first impressions are not misleading; the Brouwer Theorem, together with its proof, presents mathematicians and philosophers of mathematics with an antimony, one that can only be resolved by a close, foundational study of the structure of the intuitive continuum.

It is no simple and uncontentious business to pronounce authoritatively where, in any body of scientific thought, natural science or mathematics ends and philosophy takes over. All serious mathematics, with set theory and category theory as premier examples, makes assumptions open to philosophical examination. In this, mathematical intuitionism is no exception. The intuitionists L.E.J. Brouwer (1881 - 1966) and Arend Heyting (1898 - 1980), perhaps like all revolutionaries, were given to philosophical rumination. In Europe during the early 20th Century, it seems that mathematical papers could be written and published in a style more verdantly literary and philosophical than is common today. These commonplaces and historical facts may have misled some into thinking that intuitionistic mathematics owes a greater debt to philosophy than its classical cousin. Yet, in truth, the intuitionism inspired by Brouwer and Heyting is mathematics first and philosophy later.

One would now be hard pressed to say what the official or intended philosophy of intuitionism is. There seems no more eagerness, among contemporary intuitionists, to embrace the philosophy once adumbrated by Brouwer than there is among contemporary set theorists to embrace the extramathematical thought of Georg Cantor. Heyting's eclectic outlook, as presented in [Heyting 1931], does not now seem an attractive option. Dummett's antirealism (see [Dummett 1977 and 2000]) remains the bare outline of a philosophy, awaiting clear specification of what is meant by suitably antirealistic 'proofs' and 'constructions.' Once those specifications are given, a further goal is still to be reached: intuitionistic logic must be proven sound and complete with respect to the antirealistic semantics of proofs and constructions, and by strictly intuitionistic means. Until such time, antirealism cannot supply intuitionism with the foundation antirealists so desire.

That intuitionism is not itself a philosophy does not mean that intuitionism never presents philosophical problems. Since intuitionism is mathematics, today's mathematics contains contradictions: there are mathematical arguments of great initial plausibility, one intuitionistic and one conventional, whose conclusions stand in straightforward contradiction. To take a famous example, conventional mathematicians claim to prove, from axioms for analysis or set theory, that some total, real-valued functions of a real variable are discontinuous on the closed interval $[0, 1]$. Those mathematicians set out from widely accepted principles governing real numbers and functions, plus a definition of continuity, and arrive at the conclusion that there are real functions over $[0, 1]$ whose graphs are broken. With more set theory, he or she thinks to prove that the discontinuous functions outnumber the continuous. At the same time, intuitio-

nists prove with all due rigor Brouwer's Continuity Theorem: that every total, real-valued function is uniformly continuous on $[0, 1]$ and, hence, that no total function is discontinuous there. The intuitionists' proof also calls upon reasonable principles about real numbers, functions and sets, and the very same definition of continuity as that commonly employed. What the conventional mathematician asserts, that very thing the intuitionist denies.

Philosophers of mathematics commonly seek to reject descriptions of this state of mathematical play as disturbingly contradictory, one in which there is real disagreement between intellectually attractive alternatives. To that end, some claim that, although Brouwer's Theorem might well contradict results of ordinary real analysis, intuitionistic mathematics cannot be an intellectually attractive competitor to its classical cousin because it is disastrously weak, relying upon a deductively puny logic. By their lights, intuitionism lacks one of the principal virtues of a successful scientific theory, that it yield enough consequences of the right sorts. Famously, there is a range of conventional consequences thought to be of a 'right' sort that the intuitionist cannot obtain. Those are the intuitionistic equivalents of the *tertium non datur* or **TND**,

$$\phi \vee \neg\phi.$$

Conventional mathematicians deem **TND** mathematically essential. Exercising his gift for memorable hyperbole, David Hilbert (1862-1943) once exclaimed, "Taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists." [Hilbert 1927 476] Both parties to the dispute agree that, without this assumption, one cannot prove such mainstays of conventional real analysis as the Intermediate Value Theorem: that the graph of a continuous function whose value is strictly positive at the left end of a nontrivial closed interval and strictly negative at its right end has to cross the x -axis at least once in the interval. Critics of intuitionism maintain that the omission of **TND** drags intuitionism below the level of acceptable theory.

In decrying supposed weakness in intuitionistic mathematics, commentators are not drawing a purely syntactical comparison between formal systems. Hilbert was not distraught merely because the intuitionist, in some formalism for propositional logic, could not produce a string or string type of this kind

$$p \vee \neg p.$$

That this formula is underivable does not show that the intuitionist has missed a truth. It is no barrier to the success of conventional logic that,

in systems with \rightarrow and \neg as sole connective signs, the above-displayed shape is absent from the list of formal validities. Of course, there could be a dispute among mathematicians over the relative expressive powers of two competing theories, but that is not the issue here.

The trouble over **TND** has to do with propositions or truths expressed. The intuitionist maintains that the scheme **TND** captures no truth or validity he or she has overlooked. The intuitionist proves that **TND** is invalid using strictly mathematical assumptions. For example, Brouwer's Theorem itself entails the failure of **TND**. Were **TND** generally valid, every real number would be rational or irrational. It follows by function comprehension that the salt-and-pepper function, outputting 1 on rational numbers and 0 on irrationals, would be total and real-valued. However, it is plainly an everywhere discontinuous function. Hence, the *tertium non datur* is not valid, QED. In consequence, the intuitionist feels herself under no more obligation to accept the validity of **TND** than she is to accept the validity of other plainly invalid principles, for example,

$$p \vee \neg q.$$

To harp on the intuitionist's inability to prove **TND**, or to imply that it is merely an unfortunate and remediable lack, would be to ignore or prejudge the central issue presented by the intuitionistic version of the mathematical facts. That **TND** is invalid, like the Brouwer Theorem, is a fact of the latter kind.

The failure of **TND** does not appear to create an insurmountable obstacle to the progress of intuitionism, even in the realm of applied mathematics. Errett Bishop (1928-1983) and coworkers in constructive mathematics have demonstrated that, although Brouwerians cannot claim the Intermediate Value Theorem, they can derive valuable consolation prizes. They proved that reasonable approximations to the Theorem are obtainable intuitionistically, among them the theorem that, for f continuous over $[a, b]$ with $f(a)$ negative and $f(b)$ positive, and for any natural number n , there is a real number c_n in $[a, b]$ for which the absolute value of $f(c_n)$ is less than $1/n$. [Beeson 1985 11-12] Logicians know this as an instance of a general phenomenon: every conventional theorem Φ of set theory or arithmetic admits a reformulation Ψ such that Φ and Ψ are equivalent conventionally but Ψ is a theorem of intuitionistic mathematics. Recently, the reach of intuitionistic mathematics received further confirmation. Geoffrey Hellman had challenged constructivists to prove Gleason's Theorem, which he believed foundational to quantum mechanics. [Hellman 1993] Douglas Bridges and Fred Richman responded by proving that the Gleason Theorem (not a reformulation, but

the statement as Gleason originally had it [Gleason 1957]) is a theorem of Bishop-style constructive mathematics and, hence, of intuitionism. [Richman and Bridges 1999]

No one should go away thinking that charges of weakness and inadequacy can be lodged only against intuitionism. Intuitionists have discovered and mapped mathematical worlds that are hidden from conventional sight. Among these are nontrivial standard models for Alonzo Church's (1903-1995) untyped λ -calculus, important to the semantics of programming languages. The untyped λ -calculus is a formalism for function definition and application; its leading idea is Church's type-busting conception of a collection of mathematical items that are both functions and arguments to those functions simultaneously. Type and size restrictions enforced in conventional set theory prohibit any standard conventional model of this notion from existing; Cantor proved that no set with at least two members can stand in one-to-one correspondence with the collection of all its endofunctions. In 1969, Dana Scott showed logicians how to build attractive and informative models of the untyped λ -calculus [Scott 1973], but these are nonstandard in that only continuous functions are allowed. Here, as elsewhere, the intuitionist can do his or her conventional colleague one better. Within an intuitionistic theory of sets, there are significant domains of entities that stand in one-to-one correspondence with the space of all functions on them. [McCarty 1983] In these function domains, every entity can be both a function and an allowable input to those very functions: for any element f of the intuitionistic structure, $f(f)$, the application of f to itself, yields an output value. The intuitionist realizes Church's vision with perfect accuracy; the conventional mathematician can, at best, approximate it.

Others endeavor to defuse the mathematical standoff between intuitionistic and conventional mathematics by arguing that scholars have misinterpreted the statements of intuitionistic mathematicians when they claim that Brouwer's Theorem contradicts a result of classical real analysis. Although he may no longer endorse the relevant arguments (cf. the introduction to [Tait 2005]), William Tait once maintained that the claims of intuitionistic mathematics do not really contradict those of conventional mathematics, because the former is in fact a part of the latter. [Tait 1983] Were that the case, intuitionistic mathematics would be consistent with classical mathematics, provided that classical mathematics is itself consistent. One of Tait's arguments for his conclusion seemed to rely upon the following assertions.

The distinction between constructive and classical rests solely on what principles are admitted for constructing an ob-

ject of a given type. And since all constructive principles are classical too, constructive mathematics is a part of classical mathematics. A classical proof [that] $a:A$ [a is an object of type A] is constructive just in case it is obtained using only certain principles of construction. [Tait 1983 182-183]

‘Constructivism’ is said in many ways: strict finitism, Hilbertian finitism, Bishop-style constructive mathematics, the Russian constructivism of Markov and followers, ultra-intuitionism, Erlangen constructivism, Russellian predicativism, among others. Certainly, adherents to these outlooks adopt principles for constructing typed objects, but not all such principles are classical. In Russian constructivism, Church’s Thesis rules the roost: every total function with natural number inputs and outputs is general recursive. On that approach, to construct a higher-order functional of type $[N \Rightarrow N] \Rightarrow N$, that is, a map taking total numerical functions into the natural numbers, it would be sufficient to specify a unique natural number output for each total recursive function input. This is not classically correct; the global behaviors of classical functionals of type $[N \Rightarrow N] \Rightarrow N$ are not in general determined by their actions on the total recursive functions.

Furthermore, it is hardly the case that the sole distinction between constructive mathematics, including intuitionism, and classical mathematics is “what principles are admitted for constructing an object of a given *type*.” There are far more things in Brouwerian heaven than are dreamt of in a philosophy of types. Often, a type is an abstract, conglomerate entity over which the Axiom of Choice reigns. Because **TND** is intuitionistically invalid and the Axiom of Choice implies **TND** [Diaconescu 1975], there must be nontypes aplenty. In particular, there are nontrivial subtypes and such quotients of types as the domain of all partial recursive functions, which is a quotient of the set of natural numbers. Kleene’s number realizability provides an interpretation of intuitionistic set theory in which the Axiom of Choice fails over that very domain. [McCarty 1983] [Beeson 1985]

Many standard formalisms associated with intuitionistic mathematics, among them Heyting arithmetic and intuitionistic Zermelo-Fraenkel set theory, are subtheories of their classical correlates, in this case, Peano-Dedekind arithmetic and Zermelo-Fraenkel set theory, respectively. That familiar metatheoretic fact in no way entails that intuitionistic mathematics is a part of classical mathematics. The theorems of intuitionistic mathematics should not be identified, even extensionally, with the truths expressed by the formulae of any formal system or systems on their

standard interpretations. The reasons for this are well known. First, a mathematical subject, like topology or arithmetic, is neither a formal system nor a battery of them. While formal systems are individuated and articulated by formal languages and their grammars, mathematical subjects are individuated and articulated according to the desires of mathematicians, especially the authors of mathematical textbooks and monographs. For one thing, the Turing machines that control the operations of formal systems know nothing of the importance or unimportance of any mathematical results their symbolic theorems may represent. Second, intuitionistic mathematics is not provably complete with respect to the familiar formalisms deemed to represent it, even when it comes to propositional logic. Under number realizability, there is a simple propositional formula in two variables that is valid but has instances unprovable in intuitionistic Zermelo-Fraenkel. [McCarty 1991]

Logicians also know that certain powerful intuitionistic systems, including some that derive formalizations of Brouwer's Theorem, are sound with respect to topological, realizability and category-theoretic interpretations. [Troelstra and van Dalen 1988] Thus interpreted, all theorems of those systems can be taken to yield conventional truths (provided the metatheory is conventional) concerning topological spaces and continuous functions, or sets with recursive structures, as the case may be. For example, under realizability, **TND** fails because it entails the solvability of the halting problem. Were it supposed that one or another of these interpretations tells the world what intuitionism really means, the truths yielded on those interpretations by intuitionistic formulae would be (parts of) intuitionistic mathematics and conventional mathematics at one and the same time. Hence, a goodly portion of intuitionistic mathematics, viewed in this manner as a set of conventional truths, could not contradict true classical mathematics. On this proposal, Brouwer's-Continuity-Theorem-so-construed would be what the Theorem truly says, and would not stand in logical opposition to any correct finding of ordinary real analysis.

It is unfortunate for this suggestion that no one seems able convincingly to say which of the vast range of available interpretations for intuitionistic formal theories is guaranteed to provide the supposedly true conventional meanings of those theories. Within realizability, Brouwer's Theorem gives a version of the Kreisel-Lacombe-Schoenfield-Ćeitin Theorem for effective operations [Beeson 1985 61-64]; within the topological model for analysis devised by Scott [Scott 1968], it induces a very different statement about sheaves over Baire space. It would seem that both cannot be candidates for the one meaning of the original theorem. It remains to be determined which (or which of the infinite variety of

interpretative alternatives to them) captures the real and unequivocal meaning of Brouwer's great discovery.

The supply of truths obtained by filtering formal intuitionistic theorems through the aforementioned interpretations does not generally amount to any recognizable branch of conventional mathematics. Those interpretations often show little respect for the significance intuitionists attach to their discoveries. Sometimes, the conventional truths expressed under interpretation by important intuitionistic theorems turn out to be singularly unimportant. In a topological model over Sierpinski space, the failure of the validity of **TND** registers the triviality that the Sierpinski topology is not discrete.

The formulae logicians find holding, thanks to conventional metalogic, in a topological or realizability model, those comprising the 'internal mathematics' of the structure, do not always correspond to intuitionistic statements or reasonable facsimiles thereof. In consequence, they seem to offer no serious candidates for the true meanings of the sayings of intuitionists. Assume that the model giving us the official conventional meanings of intuitionistic statements is Scott's model for analysis [Scott 1968]. In that structure, Brouwer's Continuity Theorem holds, as do all true formulae of conventional first-order arithmetic, when the metalogic is classical. This fact means that a proponent of the suggestion under review has some explaining to do. To see this, let $T(x, y, z)$ be the Kleene T -predicate internal to the elementary arithmetic of Scott's model. In that structure, the true statement

$$\neg\neg[\exists y.T(x, x, y) \vee \neg\exists y.T(x, x, y)]$$

is presumably the meaning, both intuitionistic and conventional, of an instance of an intuitionistic theorem of logic, namely,

$$\neg\neg(\phi \vee \neg\phi).$$

The substatement one gets by eliding the leading double negation,

$$\exists y.T(x, x, y) \vee \neg\exists y.T(x, x, y),$$

is also true, and is the Scott interpretation of the corresponding instance of $\phi \vee \neg\phi$. Although, in conventional logic and in the Scott structure, the latter formula is fully equivalent to the former, it represents no truth of intuitionistic mathematics. (It would be taken by Russian constructivists to make a plainly false claim.) There is then nothing in the presumptive meanings of the two statements and the mathematics of Scott's structure to explain these facts, to explain why the intuitionist would happily

endorse the first formula and fervently refuse the second. In this respect, Scott's model is no special case; similar examples can be found for realizability interpretations.

Lastly, one worries that this suggestion – that intuitionistic mathematics is identifiable, in whole or in part, with the interpretation of a formalism – is born from an equivocation involving the word 'interpretation.' The interpretations logicians supply for metalogical purposes need not be and generally are not interpretations in the everyday sense of the term. Interpretations in the latter sense serve helpfully to illuminate the usual meanings of one or more sayings; meanings that might otherwise be dark are brought into interpreted light. The interpretations of logicians, on the other hand, may have little or nothing to do with the literal or metaphorical meanings of most words. When one interprets as a logician, one is required to preserve no part of the interpreted saying that is nonlogical; only the logical words, the 'if's, 'and's and 'all's, need to retain their familiar semantical values. For example, if there are an infinite number of disposable razors, a logician can employ the countable Löwenheim-Skolem Theorem to cull from the world of disposable razors and their properties a perfectly appropriate logical interpretation of all the axioms and theorems of Zermelo-Fraenkel set theory. It hardly follows that set theory can be interpreted, in any ordinary sense, as making spectacular claims about lots and lots of disposable razors.

Some believe that the contradiction posed by intuitionistic analysis can be avoided by appeal to the special constructive meanings of the logical signs. They would argue that, as a matter of literal, non-interpreted fact, what the intuitionist says in asserting a mathematical sentence Φ usually means something quite different from the same Φ issuing from the mouth of a classical mathematician. They would allow that the intuitionist's Φ may be syntactically indistinguishable from that of her conventional counterpart, that the two mathematicians may advance syntactically identical definitions of 'number,' 'set,' 'function,' 'topological space,' and so forth, and that the two will stand in perfect syntactic accord over the truth rules that govern their quantifiers and connectives, e.g., that conjunctions (ϕ & ψ) are true just in case both ϕ and ψ are true. In the face of this large measure of unanimity, the philosophers in question still insist that the connectives and quantifiers of the intuitionist carry meanings distinct from those attached to the utterances of the classical mathematician, the intuitionistic meanings being given by a proof-theoretic semantics alternative to standard, truth-conditional semantics. If justified, such a move might draw the sting of paradox from Brouwer's Continuity Theorem. Should the intuitionist's

logical signs bear meanings distinct from the same symbols in conventional mathematics, even if the definitions of ‘real number,’ ‘real-valued function’ and ‘continuous’ are identical in form across both brands of mathematics, the statement proved by Brouwer would not necessarily be a denial of the familiar theorem that total, discontinuous, real-valued functions exist over $[0, 1]$.

According to proponents of proof-theoretic semantics, the meaning of a sentence should not be given in terms of truth conditions, realistically conceived as conditions that could obtain regardless of anyone’s being able, even in principle, to recognize their obtaining. Instead, sentence meanings should be specified by giving proof conditions, circumstances under which a sentence would be recognizably proven. An intuitionist’s sentence would then be considered true if and only if a proof of it is available. A sentence would be false if it is recognizably disprovable — perhaps by a plain reduction to an evident absurdity. It is believed that, on this semantical account, some well-formed mathematical sentences will fail of both truth and falsity. At present, Riemann’s Hypothesis (**RH** for short) would be conveyed by such a sentence. There now exists neither proof of **RH** nor disproof, and mathematicians are aware of no real prospect for altering that circumstance. Therefore, neither **RH** nor its negation would currently be true with respect to proof-theoretical semantics, and the instance $\mathbf{RH} \vee \neg\mathbf{RH}$ of **TND** fails.

Promising as it may seem, this effort to avert contradiction does not succeed. Just as in conventional mathematics, proofs in intuitionistic mathematics rely ultimately on basic principles, perhaps the principle of complete induction for the natural numbers, or the union axiom for sets, or (a form of) Zermelo’s axiom of separation. Intuitionists rightly take these and other fundamental principles to be true, they construct proofs with them, and they come up with noncircular, reasonably convincing arguments for endorsing them. The crucial point is that these reasonably convincing arguments are, one and all, plausibility arguments or intuition pumps, not proofs in full regalia. They are intuitionistic analogues to the kinds of hand-wavings conventional mathematicians employ when they first introduce their students to the Zermelo-Fraenkel axioms. The intuitionists feel fully justified in asserting their basic principles, just as classical set theorists feel justified in asserting the Union Axiom. Intuitionists cannot, and usually do not attempt to, prove the basic principles. Intuitionists know they are true. After all, that was one of the original, great ideas of Brouwer’s intuitionism: there are fundamental properties of important mathematical structures, with unlimited induction on the natural numbers and forms of set comprehension as paradigmatic ex-

amples, that are truths of neither language nor logic, but are recognized nondeductively to obtain. These principles are true, but proofs of them are not available. This realization is required for the success of intuitionistic mathematics.

Incidentally, a die-hard advocate of proof-theoretic semantics can hardly now respond that, when it comes to basic principles, the intuitionist postulates or assumes the existence of proofs for them. Once we allow ourselves to shore up claims by postulating their proofs into existence, we can happily postulate a proof for **RH** or **TND** and have done with it. A similar response is available to anyone who suggests that the extension of ‘intuitionistically provable’ be enlarged by stipulation to include the sorts of nondeductive justifications intuitionists commonly provide. It would then have to be explained what objection the intuitionist could make to a classical colleague who decided to stipulate thusly that **TND** is henceforth provable intuitionistically.

The efforts here assayed for disarming the clash between intuitionistic and conventional mathematics fail. At the heart of mathematical analysis lies contradiction, not equivocation. The theorems of intuitionism should be taken at face value: when the intuitionist asserts that a number specified by ρ exists or that a function defined by Φ is continuous, he or she is making the very same statement as a conventional mathematician who asserts that number ρ exists or that Φ is continuous. And the intuitionist is contradicting mathematicians who insist that ρ does not exist or that Φ is actually discontinuous. Intuitionists, who dispute the truth of various theorems of ordinary mathematics, should be allowed their straightforward, mathematical say, without re-interpretative interference. It’s time to face the foundational music and recognize that the steps in Brouwer’s argument for his Continuity Theorem are cogent and its ultimate assumptions are plausible. Each form of mathematics, conventional and intuitionistic, offers its own persuasive depiction of the primary object of analysis, the intuitive continuum, and not a sheaf-theoretic *Doppelgänger* or a quasi-continuum in one or another universe for a set theory relativistically construed.

Perhaps the opposition between intuitionistic and conventional mathematics is the upshot of a subtle mathematical error hidden within the presuppositions of conventional analysis. To locate that error would be a project for foundational study. To take up that project is to rejoin a tradition made venerable by the thought of Eudoxus, Archimedes, Newton, Cantor, Dedekind, du Bois-Reymond, Hilbert, Brouwer and Heyting. Intuitionists remain more than a little hopeful that this project will vindicate the ideas of du Bois-Reymond, Brouwer and Heyting, and

expose the flaws in the designs of their opponents Cantor, Dedekind and Hilbert.

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