



Philosophia Scientiæ

Travaux d'histoire et de philosophie des sciences

16-1 | 2012

From Practice to Results in Logic and Mathematics

Practice, Constraint, and Mathematical Concepts

Mark C. R. Smith



Electronic version

URL: <http://journals.openedition.org/philosophiascientiae/710>

DOI: 10.4000/philosophiascientiae.710

ISSN: 1775-4283

Publisher

Éditions Kimé

Printed version

Date of publication: 1 April 2012

Number of pages: 15-28

ISBN: 978-2-84174-581-4

ISSN: 1281-2463

Electronic reference

Mark C. R. Smith, « Practice, Constraint, and Mathematical Concepts », *Philosophia Scientiæ* [Online], 16-1 | 2012, Online since 01 April 2015, connection on 03 November 2020. URL : <http://journals.openedition.org/philosophiascientiae/710> ; DOI : <https://doi.org/10.4000/philosophiascientiae.710>

Tous droits réservés

Practice, Constraint, and Mathematical Concepts

Mark C. R. Smith

Queen's University (Canada)

Résumé : Dans cet article je propose d'exprimer et de défendre une conception des pratiques et du domaine de discours mathématiques qui soit sensible, d'une part, au pluralisme des relations entre pratiques inférentielles et intérêts, et d'autre part, à la structure objective et déterminante des concepts mathématiques. J'ébauche tout d'abord une caractérisation générale des pratiques, pour ensuite préciser certains phénomènes propres aux pratiques mathématiques. Suit un recensement des idées qui se dégagent des arguments pluralistes, et de celles qui sont à retenir. Mais je défends par la suite la nécessité d'une forme de réalisme mathématique, qui toutefois ne peut être le réalisme d'objets que prônent les partisans de l'argument d'indispensabilité. Je défends plutôt un réalisme des concepts, soutenu par ce que je baptise l'« argument de la contrainte ».

Abstract: In this piece I articulate and defend a conception of mathematical practice and mathematical subject-matter, which is responsive both to a sensible pluralism concerning the connection between inferential practices and guiding interests, on the one hand, and to the objective content-determining structure of mathematical concepts on the other. I begin by sketching a general characterization of practices themselves, and by specifying some of the unique features of mathematical practices. An examination of inferential pluralism follows, and some insights of pluralist arguments are retained. But I argue further for the requirement of some form of mathematical realism, though the object-realism of the indispensability argument is assessed and rejected. My positive proposal argues for a form of concept-realism, which is established by what I call the "argument from constraint."

Introduction

A good deal of what goes on in our ordinary lives and in our reflective theoretical moments is shaped by an array of practices of various kinds—conceptual, cultural, technological, and others—and philosophy of late has become increasingly sensitive to these, in particular to their role in shaping the substructure of normative thought in contexts of both practical and theoretical reasoning. How we think emerges constitutively out of what we’re up to, interested in, find salient, or simply need in virtue of being human. In the specific case of mathematical reasoning, we might regard the very existence of multiple, mutually incompatible derivational systems as evidence for a kind of ‘practice pluralism’, the thesis that different forms of conceptual life—so to speak—are bound to yield different results according to their own *internal* constraints of reasoning, absent a system-*external* source of norms of mathematical thought. The pluralist idea, loosely stated, is that in the context of pure mathematics, it is practice which determines the content of at least most mathematical concepts. But a large part of my argument here will be that this view is false, and I will both motivate the negative or critical aspect of my view, and suggest an alternative in the light of a novel argument.

Mathematical practice has some features which are distinctive. In general a mature practice or cluster of practices can be individuated thanks to at least these three aspects.¹ The first is that a practice is individuated by the network of concepts in which that practice finds its place as something meaningful and recognizable as such among a community. Call this the interpretive structure into which a practice must fit. To illustrate: since I have no interpretive structure for telling the difference between, say, people really playing cricket and people going through some joking simulacrum of cricket, I have no serious way of telling whether or not the practice of cricket is being embodied or carried out by the group I’m watching. My lacking an interpretive structure, such as the application criteria for the concepts of *run* and *wicket* and so on, not only makes me an outsider with respect to the practice, but makes me incapable of distinguishing it from an imitation or a close, but distinct, relative. Secondly, a mature practice is something which must affect the mental states of those who are undertaking it. In order for a practice to be recognizable as cricket or waltzing or doing group theory, it must be part of the participants’ self-conception that they are engaged in that very activity: there must be first-person acknowledgement available. And thirdly, a practice must be ‘something to which we advert in describing an activity in certain terms’, as Thompson [Thompson 2008, 199] puts it: that is, there must be second- and third-personal acknowledgement available—for instance of an activity as counting as waltzing—in addition to the first-personal self-conception.

1. I borrow these three criteria loosely from Michael Thompson [Thompson 2008, 199], though the use I make of them is entirely different.

The multiple branches of mathematics share these characteristics just as much as do the practices of making a soufflé, presiding over a trial, or investigating the anatomical structure of the fins of certain fish. In order to see what is most intriguing about mathematics understood as a collection of practices, though, we can make a further distinction among practices which are non-truth-seeking (such as making a soufflé), and practices like anatomical investigation which *are* truth-seeking with respect to a given subject-matter. Mathematics both pure and applied is a practice of the latter kind. But it has the further remarkable feature that, while it is a cluster of mature truth-seeking practices, its content is (generally) not available for empirical inspection and its truths are not (in the general case) open to confirmation in the face of experience.

And so my interest in what follows is in asking what the results of mathematical practices, so characterized, can tell us about the distinctive target-domain of (some portion of) mathematics. My specific thesis will be that mathematical results—at least in some significant range of cases—cannot have their content fixed by the practices that engender them, but rather that the direction of fit is closer to the reverse: something anterior to the practice is what ratifies the practice itself. I will, therefore, be arguing for a kind of realism, one which, as I shall explain, concerns *concepts* rather than objects, and which I will try to make persuasive in light of a specific, familiar, and quite interesting example.

The rest of the paper is organized as follows. In section 2, I will broach the contrast between practice monism (as embodied by Michael Dummett, on whose view there is one single correct canon of inferential procedures) and practice pluralism (as represented by Hartry Field), and argue in favour of retaining some insights from each of their quite different views. Part of my argument here will be that while practice pluralism is closest to the truth, pluralism itself offers a series of examples of how practices adjust to target-domains and have their content fixed by those domains, rather than the reverse. But what, precisely, is involved in speaking of a mathematical ‘target-domain’? This is the question explored throughout section 3. In 3.1, I make the preliminary case that realism of one sort or another has to be endorsed, but, *contra* the indispensability argument discussed in 3.2, it cannot be a realism *vis-à-vis* objects. In 3.3, I flesh out a fuller picture of my rival argument—the argument from constraint—to motivate the beginnings of a more satisfactory, practice-motivated realism. Finally, in section 4, I draw some conclusions and make explicit the connection between practices, constraint, and the insights retained from the second section.

1 The practice of inference: harmony and pluralism

Alternative logics, embodying alternative models of what inference can or should be, are designed in such a way as to preserve different features for varying purposes. While classical logic has truth-preserving inference in view, combined (perhaps necessarily) with a realist conception of semantic values for mathematical propositions, intuitionistic strictures on logic are meant to preserve epistemic *warrant* in inference, while a paraconsistent system which denies ‘explosion’ or *ex falso quodlibet* might conceivably be suitable for an account of inferences in reasoning about the contents and structure of fictional worlds (and there may be other uses of paraconsistency as well).

But standing against this first blush of a pluralist vision of inferential practices, we find Michael Dummett (see especially [Dummett 1978]). The elements of his argument are familiar but bear repeating, for they culminate in an insight about *harmony* which I think is worth retaining. Begin with an everyday sort of example about the practice of inference. Imagine that we are speaking of someone who is now dead, and we wonder if she was courageous. Let’s say that she was never in danger, never faced a situation in which she might be called upon to display courage or show herself to be a coward. Is it the case that there is some truth ‘out there’, some verification-transcendent truth, to the proposition that she was courageous? Surely not: surely it is the case that ‘She was courageous’ is neither true nor false. It is *not* that she really was one way or the other and we’ll never know which: the issue is that it seems that this proposition does not have a truth value at all. In Dummett’s view, this familiar sort of observation is an indication that we should not tether meaning to bivalent truth conditions, because the sentence ‘She was courageous’ is both meaningful and deprived of a truth value. We can say:

- (1) It is not the case that she was not courageous.

But that does *not* entail:

- (2) She was courageous.

What we know is what would have had to have been the case in order for us to be entitled to assert ‘She *is* courageous’ or ‘She showed courage on that occasion’, but now that she is dead we should not suppose that there is some truth, or some falsehood, about her that just happens to be forever outside our epistemic reach. Rather, we know what it would have been like to be *entitled* to the assertion, and so it is meaningful, but that does not license the inference from (1) to (2). That inference would be licensed, by double negation elimination, on a truth-value-realist’s semantic picture. But there is certainly something deep and important about Dummett’s contention that the inference

is suspicious. Surely, she was neither the one nor the other. But we speakers of English are comfortable with explaining what kinds of (counterfactual) circumstances *would have entitled us to say* that she was either courageous or cowardly, in a certain situation. I think that the resonance of the argument comes from an expectation of *harmony*, a property we expect to find among our inferential practices and the structures they invoke. Understanding mathematical language and thought, with their putative entities and structures and inferences, involves just the same machinery as any other portion of thought. These involve the ability to fit ‘molecular’ semantic pieces into a wider network of features, such as the evidence on which some proposition rests and the consequences that flow from that proposition. This is what the harmony at issue involves.

On this view, understanding any proposition p involves: (i) mastering at least some of its appropriate contexts of use in the light of what other speakers recognize as appropriate, (ii) the ability to draw out some of what follows from it, (iii) the ability to transpose it into another appropriate context (e.g. when a child hears a metaphor for the first time, and can display her understanding or failure to understand that metaphor in her subsequent transposition of its use to fresh contexts which are either appropriate or inappropriate), and (iv) being able to adduce warrant for it. I know nothing about industrial chemistry, but I know that gold dissolves in cyanide (and indeed that ‘Gold dissolves in cyanide’ is $T \leftrightarrow$ gold dissolves in cyanide). But if I go to a meeting of industrial chemists, they will soon see that I don’t *understand* this proposition in any real way because I can’t adduce evidence for it, I can’t draw any consequences from it, and I can’t transpose it to any other relevantly similar circumstance. I have no network into which to fit p : it has no harmonious position in my cognitive equipment. In fact I have no interpretive structure which would afford me the chance of being a participant in the practices that people who really understand the proposition are capable of. Mathematical propositions, like any other cluster of propositions, are located in a network of evidence (warrants) and consequences, which I need to master in order to master the relevant portion of the language. I need substantially more than truth conditions for just p alone: I also need its *fit* into a network of relevant collective practices of use, giving and taking evidence, and consequence-drawing. The significant insight to retain from Dummett is that engagement in mathematics (or of any other portion of thought and language) is a skillful deployment of practical abilities that cannot be represented as piecemeal, proposition-by-proposition agglomeration.

But Dummett also defends a further thesis: that the right logic should embody the single correct semantics, and so that we cannot ratify (for example) DN-elimination in the logic, by the argument from the reflection on courage given above. He defends the view that there is a unique *correct* logical canon. We might style him a ‘practice monist’ in this respect. There is, how-

ever, an apparent tension between the monistic thesis concerning inference, and the harmony thesis, since if it's the case that a practice constitutively demands a harmonious interpretive structure, and given that classical mathematics grounded in classical logic is one of the most robust interpretive structures there are, then the reformist project of monism conflicts directly with mature, robust and harmonious practices. Since the proof-strategy of DN-elimination is, furthermore, so pervasively operative over mathematical concepts that are very robust indeed, and since it allows for so many transitions among robust structures, there is very good reason to ratify it. Because such inferences are authorized in number theory, for example, the transposition of a (perfectly legitimate) constraint from inferences about one target-domain—such as states of character like courage—onto a quite different target-domain are unwarranted, and jar with the expectation of harmony.

A rather different position *vis-à-vis* inferential practice in mathematics is defended by Hartry Field (see, for instance, [Field 1998]), in whom we find a strong ‘practice pluralism’. His famous slogan is that mathematics need not be true to be good. Many ordinarily take truth to be a non-trivial property of the class of uniquely warranted mathematical propositions by the strictures of some monolithic canon. Not so for Field. The reasoning is as follows. There are many canons, many alternative logics, and each one will deliver some different inferences. A logical canon is like a licensing bureau, and different wickets in the bureau will stamp your license in different ways to ratify some inferences and prohibit others. The classical bureau will license the inference from (1) to (2), but the intuitionistic bureau won't allow it to go through. The classical bureau will license you from $(p \text{ and } \sim p)$ to q , but the window with Graham Priest behind it (see [Priest 2006]) will let p and $\sim p$ both be true and not license the inference to q , as he argues that we should not ratify *ex falso* (or *ex contradictione*) *quodlibet* in the general case. Paraconsistent logic is comfortable with inconsistent arithmetic (e.g. [Mortensen 1995], [Mortensen 2009]), while to classical logic, inconsistency is fatal. (Frege learned this the hard way.) But now, we might ask in pluralist spirit, on what ground would you stand from which you could further say ‘Classical logic is *true*, and a paraconsistent system is *false*’, or vice versa? There is no such ground. Correctness, in the case of mathematics, must just be consistency with the principles of a canon, some canon or other. The only grounds for choice between canons, for choice as to which window in the license bureau we walk up to, are either aesthetic or pragmatic. Choosing between logics is not a matter of logical choice obedient to some prior standard, nor a matter of discerning which is true. It might be a matter of choosing what's more beautiful, or picking what better serves the interests of our empirical science: in essence, the choice of interpretive structure, the selection of an appropriate practice in which to engage, is one which is guided by the aims at hand, guided by the target-domain. But how are we to characterize that domain, and the results about it?

2 The argument from constraint

The indispensability argument for mathematical realism, due in large part to Quine and Putnam, is widely held to be the most promising candidate for establishing mathematical realism concerning the mathematical target-domain, if such realism can be established at all. It's also an argument that begins with our practices as they weave together mathematics and natural science into a conceptual whole. Yet it suffers from some significant problems. In the following subsections I will begin by offering a different, and novel, argument for realism concerning mathematical concepts, contrast it with the indispensability argument, and reply to some potential objections while fleshing out the argument further.

2.1 The argument in brief

It goes like this. One of three familiar ancient Greek construction problems in geometry is the problem of squaring the circle (the two others are trisecting the angle and doubling the cube). The problem is this:

(SC) Using only an unmarked straight-edge and compasses, and given a circle of area A , construct a square having that same area A .

Greek geometers struggled with this problem for a long time, as did many mathematicians until the 19th century (mathematicians were similarly exercised by the other problems too). It turns out that the construction demanded in (SC) is, in fact, impossible, and we can prove it.

Suppose a circle of radius r . The area of a circle A_C is given by the formula $A_C = \pi r^2$. Now, let $r = 1$, to give us an arbitrary unit length. The area A_S of a square of side a is, of course, a^2 . This means that to construct $A_S = A_C$ we would need to construct a square with side $a = \sqrt{\pi}$.

This is what's not possible. π is a transcendental number: this means that it does not satisfy a polynomial equation (i.e. one of the form $a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$). Now, polynomial equations operate exclusively with the functions of addition, subtraction, multiplication and exponentiation: these are the operations that can be geometrically constructed, that is, carried out in principle with the tools set in (SC). But π does not satisfy any procedure that involves just these operations: it is transcendental (the proof was given by Lindemann in 1882 [Lindemann 1882]). The upshot is that there is no constructive geometric 'performance', so to speak, by which we could accomplish the task set in (SC). The transcendence of π means that we cannot turn one structure-type into another by a certain pathway. So a purely mathematical property furnishes an *explanation* of why something is impossible, an explanation of a constraint on the possibilities that are within the ambit of our practice.

In order better to see the force of the argument from constraint over the indispensability argument ('IA', for short), I shall consider two points of contrast. The first has to do with IA's ultimate inability to derive its target conclusion on the basis of premises that require mathematical objects' explanatory role to *match* the role of empirical postulates. The second point raises the question of how one might argue for the actual satisfaction of mathematical truth conditions. I raise each of these in turn.

2.2 A mismatch of roles

The general form of the indispensability argument for the existence of specifically mathematical abstract objects invokes their indispensability for empirical science. There is in fact so much mathematics included in the interpretive structure of the natural science that the structure would crumble, and the natural scientific practices become unintelligible, if the mathematics were removed. I borrow the following formulation of the argument's principal thesis from Colyvan:

If apparent reference to some entity (or class of entities) ξ is indispensable to our best scientific theories, then we ought to believe in the existence of ξ . [Colyvan 2001, 7]

The idea might be cast like this: apparent reference, combined with indispensability (in some sense to be made more precise) in the formulation of a body of theory which we know on independent grounds to be true, *secures* reference. Commitment to mathematical objects, by these lights, is the very same sort of thing as commitment to quarks and other unobservable entities. If 'quark' appears to refer, and we cannot achieve the same scientific success without reference to quarks as we can with such reference, then it is rational to believe that 'quark' refers, even if it is only a provisional belief. In a similar vein, if 'the real line' appears to refer, and we cannot achieve the same degree of explanatory success in physics without the real line as we can with it, then it is rational to believe that the expression refers. IA claims that mathematical objects fill the same explanatory role as physical unobservables.

The argument's fulcrum is the extent of the empirical support we can bring on behalf of this thesis: that mathematics cannot, without loss of explanatory content and disintegration of the interpretive structure, be paraphrased out of empirical science. An IA of this kind is, then, an empirical hypothesis, to be tested like any other against experience. In that vein, in a piece on behalf of mathematical objects' indispensability in natural science, Alan Baker [Baker 2005] raises the very interesting example of North American periodical cicadas and the resources that need to be brought to bear on the explanation of their life-cycle. Briefly, the case is this: in each of three species of cicada of the *Magicicada* genus, 'the nymphal stage remains in the soil for a lengthy period, then the adult cicada emerges after either 13 years or 17 years depending on

the geographical area' [Baker 2005, 229]. That the life-cycle should fall into a prime-numbered-year sequence is curious, and wants explaining. Baker canvasses two rather different biological explanations of why this is an evolutionarily advantageous sequence, both of which, despite their differences, invoke the number-theoretic fact that prime periods minimize intersection compared to non-prime periods. (Exactly what they minimize intersection *with* is open to interpretation, but Baker's discussion of the phenomenon is uniform across both of the interpretive versions he cites: see the references in his article.) In any event, Baker writes:

[T]here are genuine mathematical explanations of physical phenomena, and the explanation of the prime cycle lengths of periodical cicadas using number theory is one example of such. If this is right, then applying inference to the best explanation in the cicada example yields the conclusion that numbers exist. [Baker 2005, 236]

Nevertheless, he continues,

Whatever cases of putative mathematical explanation the platonist might come up with, there will always be some leeway for nominalist objections since the role of mathematical posits is unlikely ever to *exactly* match the role of concrete unobservables, such as electrons. [Baker 2005, 236]

I think it's helpful to cast Baker's argument like this: since explanation is achieved by invoking a property that is instantiated by nothing at all unless it's instantiated by an abstract object, the relevant abstract object is indispensable to the explanation. A further feature of the empirical IA is that it sees its criterion of success as the ability to locate cases in which the explanation of contingent physical facts ineliminably involves reference to abstract objects, whose existence is necessary if they exist at all. This opens up an opportunity for the nominalist to drive a chisel into the interstitial explanatory space between concrete and abstract particulars, and then hammer off the *abstracta*, perhaps by invoking an instrumentalist or fictionalist strategy, for instance in the manner of Hartry Field [Field 1989], for whom the mathematics lubricates inferences that it would otherwise be too clumsy to nominalize fully, but does no more duty than that. Now, my claim is that while examples like Baker's cicadas are compelling up to a point, we are ultimately left with a mismatch of explanatory role between mathematical and physical entities, a mismatch which opponents of IA will exploit by pointing out that IA does not decide the issue. (Note that the idea here is not so much that IA is false or fallacious, but that it leaves the debate just where it started.)

But we can better make the mathematical realists' case by reorienting the focus of lines of argument from a practice-centred inference to the best explanation—in which abstract objects emerge as posits on an ontological and

epistemic par with physical unobservables—to an equally practice-centred inference that will allow us to derive the explanation of a properly physical and metaphysical impossibility. Rather than looking to what mathematical entities (of some kind to be made more precise) rule in, the argument from constraint appeals to what they rule out. Mathematical entities don't need to match the role of concrete unobservables: they do a different kind of duty, but explanatory duty nonetheless. The thrust of the argument from constraint is to reveal rational grounds to endorse the existence of mathematical entities of some kind, but grounds which, rather than appealing to an empirical hypothesis about the functional role of such entities *qua* postulates, point instead to those entities' status as necessary in circumscribing the boundaries of the possible. These boundaries are laid bare by attending to what mathematical inference reveals about constraints which are both intra-theoretic to the interpretive structure, and have purchase on the range of what can actually be done.

2.3 The argument more fully

Mark Balaguer expresses this perfectly sensible demand: 'platonists need to argue that the platonistic truth conditions of our mathematical utterances are actually *satisfied*' [Balaguer 2009, 133].² The realist needs to adduce some reasons for thinking that the conditions are satisfied *by the right sort of thing*. What might these be? Here we can make an additional point on behalf of the argument from constraint over the IA by way of an objection that teases out some important points of contrast. The following objection, analogous to Balaguer's question, might be raised against my claims for the import of constraint. If I host a party, and I run out of beer at a quarter past midnight, my not being able to serve Emily a beer at twenty past midnight is explained by my lack of beer, or by the dearth of beer in the fridge, or something along those lines. But this explanation is not an explanation that commits me to such things as dearths or lacks, as though they were objects. The proper explanation of my inability to serve enough drinks will make no mention of such an *entity* as a dearth of beer. Why suppose, then, that the explanatory resources for the impossibility of (SC) will be any different from the resources needed for the explanation of why Emily can't have a beer?

My line of reply is this. There appear to be two fundamentally different kinds of constraint invoked in these two scenarios, and two different corresponding kinds of explanation. My inability to serve enough drinks is a contingent constraint, and that constraint is explained by causal features of the situation. The relevant causal explanation will invoke contingent relations between events. The impossibility of carrying out the (SC) construction, on

2. Balaguer raises this point in respect of platonism in contrast specifically to mathematical fictionalism, but the challenge ought to generalize.

the other hand, is not an *event*, since it does not unfold in space-time, and is not open to causal explanation since the constraint expresses a metaphysical impossibility. There is no possible world in which the (SC) construction can be carried out, whereas there is obviously a possible world in which I have more beer than I have at my party in this one. We might call the mathematical explanation of the constraint an essential explanation: one that is necessary, and thus non-causal. Part of the argument from constraint's bite is that it points to a necessary property, instantiated in any metaphysically possible world whatever. This particular necessity points in turn not just to theoretical indispensability for natural science, but to metaphysical indispensability in understanding a portion of the structure of possible worlds, this one among them. The argument from constraint, then, is an argument from the requirements of essential explanation to metaphysical indispensability in the relevant interpretive structure, in which the *relata* of the explanation are not contingent events.

Now, given the phenomenon of constraint, what can the argument from that phenomenon teach us about the target-domain of the relevant portion of mathematics, such as to explain how we are able to move from practice to results concerning that domain? In other words, what underwrites the transition from practice to results in contexts such as those of essential explanation? Several candidates suggest themselves, but I will argue that only one among them is able to support the specific modal and explanatory features that must be accommodated by a suitable account.

Objects, either concrete or abstract, can be rejected as candidates for populating the target-domain. On the one hand, concrete objects are *ipso facto* contingent, and so any properties which supervene upon them, or are emergent from them, will be subject at most to physical rather than to logical or metaphysical necessity. Concrete particulars do not provide support for inferences across all possible worlds. Nor can concrete particulars, supposing them to be the relevant target-domain, *by themselves* support inferences made on the basis of mathematical induction. For instance, the unique prime factorization theorem over the integers would be falsified for any number greater than whatever number of particulars there happens to be. On the other hand, abstract objects, by the well-known arguments from Benacerraf, are equally untenable as candidates, given that a determinate characterization of any number as some particular set, or other kind of individual, is unavailable in principle [Benacerraf 1965]. Multiple and structurally indiscriminable characterizations are always at hand.

But this last observation suggests a further possible range of candidates: *concepts*, rather than objects. For concepts can of course be multiply instan-

tiated.³ If this is the case, then what is the character of the concepts at issue in the specifically mathematical target-domain of the practices involved in essential explanation? On one proposal, such concepts are to be understood as either akin to instrumental idealizations which are used in the natural sciences for modelling real systems—ideal gasses, for example, or treating fluids as continuous rather than discrete substances in fluid dynamics—or else as akin to the concepts used in fictional storytelling. On this proposal, the target-domain of mathematical practice is something *other*, which wears mathematics as a mask. It is really, on this view, the material world, or a pseudo-world of free imaginative creation, which is the target-domain. But this cannot be right, if we reflect on the phenomenon of constraint as it has been described here. For if the material world is the target-domain of mathematical practice, then there is no support for inferences concerning trans-world impossibilities; and if a pseudo-world of our imaginative creation is our real target, then the explanatory purchase of what I have called ‘essential explanation’ is lost. And so I propose that what does the work of extracting results from mathematical practice is in fact the relevant range of concepts themselves, understood as *real* at least in the sense of not being mere representations of something else (for there is nothing else that they can plausibly represent), and in the further sense that the structure of the concepts determines the appropriateness or inappropriateness of our thought about them, and the consequent practices by which we engage with them, rather than the other way around.⁴ The order of explanation appropriate to an account of mathematical correctness and the content of mathematical thought *begins* with concepts as their content and structure are worked out over time, and only then looks to whether the connected practices—practices of inference in particular, of consequence-extraction—are such as to represent the rational transitions to which we are entitled in virtue of the concept, which is gradually more fully understood. In the context of the argument from constraint proposed above, the TRANSCENDENTAL NUMBER concept determines the content of a constellation of thoughts about π and further determines the explanatory reach of those thought-contents with respect to the range of what is and what is not possible.

3. Individual concepts such as the ones expressed by proper names are an exception to multiple instantiation, but the Benacerraf argument shows that, if numbers are not individuals, number concepts are by the same token not individual concepts either.

4. In the light of these considerations I think that Daniel Isaacson [Isaacson 1994] is perhaps closest to the truth in invoking what he calls ‘concept platonism’. For concepts as determiners of thoughts do triple duty: first, unlike platonic particulars, they are multiply instantiable and thus constant across isomorphic structures that serve to fill their place-holders; second, they are constraining in virtue of possessing necessary structural relations to each other; and thirdly, following from these first two features, they are explanatory in the requisite sense of furnishing essential explanations.

Conclusion

A quick summary of the argument is in order. I suggested first that we can characterize mature practices of almost any kind along three dimensions: that they have meaning and can be individuated by virtue of being encompassed under a system of interpretive concepts; that they affect the mental state, specifically the first-personal self-conception, of participants; and that third-personal recognition of participants as such is available. But in addition to these, mathematics, among other practices, is also a discipline that seeks the truth, but the target-domain in which these truths are sought by means of mathematical practices is unique—or very nearly—in not being available for sensory examination. So what is that domain? Methodologically speaking, we need to examine some features of mathematical practices themselves, multiple practices of inference in particular, so as to corner the results in our target-domain, and here we find (retaining this insight from Field) that practices of inference are adjusted in the light of what interests happen to be guiding investigation, in the light of what kinds of results are sought. But nevertheless, the exact character of the target-domain of pure mathematics is worth inquiring into, and while I have argued that some form of realism is required, it cannot be the object-realism espoused by defenders of the indispensability argument. Nonetheless, realism is required in order that we may account for the transition from, in my example, pure mathematical practices to their genuinely explanatory outcomes (an *essential explanation*, in the vocabulary arising from the argument from constraint). I further argued that only real concepts are suited to the multi-faceted task of moving from practice to results.

Practices are always-already there, so to speak, but they are good or bad in relation to concepts antecedent in the order of explanation when it comes to phenomena such as constraint.⁵ Concepts are not generated by practice in the general case; practices are instead measured for their worth in relation to the concepts that are their target-domain. Inferential policies are responsible to concepts: hence the explanation of constraint. I have sought to motivate reconnecting the target-domain of a portion of mathematics not just with its internal structure of operation as a practice, but as a discipline whose subject matter that is importantly *practice-determining*. The argument has proceeded by pointing to what is ruled out. In my example, what is ruled out is not ruled out by practices taken by themselves, but rather by (in this specific case) the TRANSCENDENTAL NUMBER concept. Squaring the circle is not ruled out by our ways of doing, but by what we find as limits at the outskirts of our doings.

5. A related point, especially concerning infinite structures, is made in Corfield [Corfield 2010].

Bibliography

BAKER, ALAN

2005 Are there genuine mathematical explanations of physical phenomena?, *Mind*, 114, 223–238.

BALAGUER, MARK

2009 Fictionalism, theft, and the story of mathematics, *Philosophia Mathematica*, 17(2), 131–162.

BENACERRAF, PAUL

1965 What numbers could not be, *Philosophical Review*, 74(1), 47–73.

COLYVAN, MARK

2001 *The Indispensability of Mathematics*, New York: OUP.

CORFIELD, DAVID

2010 Understanding the infinite I: Niceness, robustness, and realism, *Philosophia Mathematica*, 18(3), 253–275.

DUMMETT, MICHAEL

1978 *Truth and Other Enigmas*, Cambridge: Harvard University Press.

FIELD, HARTRY

1989 *Realism, Mathematics, and Modality*, New York: Blackwell.

1998 Mathematical objectivity and mathematical objects, in *Contemporary Readings in the Foundations of Metaphysics*, edited by LAURENCE, S. & MACDONALD, C., Oxford: Blackwell.

ISAACSON, DANIEL

1994 Mathematical intuition and objectivity, in *Mathematics and Mind*, edited by GEORGE, A., New York: Oxford University Press.

LINDEMANN, FERDINAND VON

1882 Über die Zahl π , *Mathematische Annalen*, 20(2), 213–225.

MORTENSEN, CHRIS

1995 *Inconsistent Mathematics*, Dordrecht: Kluwer.

2009 Inconsistent mathematics: Some philosophical implications, in *Handbook of the Philosophy of Science*, edited by IRVINE, A. D., North-Holland Publishing Company – Elsevier, vol. 9: Philosophy of Mathematics.

PRIEST, GRAHAM

2006 *In Contradiction*, Oxford: Oxford University Press, 2nd ed.

THOMPSON, MICHAEL

2008 *Life and Action*, Cambridge: Harvard University Press.