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From Practice to Results in Logic and Mathematics

The Role of Representations in Mathematical Reasoning

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The Role of Representations in Mathematical Reasoning *

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Résumé : Cet article discute le rôle des représentations dans les preuves mathématiques. Il est suggéré ici que les représentations nous permettent de diviser une preuve en plusieurs parties plus faciles à traiter. Nous illustrerons cela avec un exemple de la pratique mathématique actuelle qui consiste à trouver la valeur d'une expression en la divisant graduellement en parties plus simples. Par ailleurs, j'explique le rôle que jouent les icônes et les indices dans cette procédure. Les icônes assurent la similarité entre l'expression et les résultats obtenus. En revanche, les indices agissent comme des indicateurs qui permettent de réinsérer les résultats aux emplacements appropriés.

Abstract: This paper discusses the role of representations in mathematical proofs. It is proposed that representations enable us to break a proof down into manageable parts. We illustrate with an example from current mathematical practice how the value of an expression is found by gradually breaking it down into simpler parts. In addition I explain which role icons and indices play in this process. Icons ensure that there is likeness between the expression and the obtained results. Indices acting as sign-posts enable us to reinsert the results at the appropriate places.

Introduction

The use of representations in mathematical reasoning is the focus of a number of papers and books in both the philosophy of mathematics and in mathematics education. Ken Manders has long stressed the use of artefacts in mathematics, see for example his [Manders 2008]. More recently the topic has been

*. I wish to thank Frederik Christiansen as well as the anonymous reviewers for insightful comments.

discussed by Grosholz in her book *Representation and Productive Ambiguity in Mathematics and the Sciences* [Grosholz 2007], where it is claimed that ambiguous representation is a prerequisite for the fruitfulness of mathematics as well as natural sciences. In [Duval 2006] it is argued that the ability to shift between different representations is fundamental for mathematical understanding. This paper seeks to clarify the role of representations in mathematical proofs. In [Carter 2010] a proof is described as using

an interplay between different kinds of representations. By representations, I mean diagrams, symbols, definitions and expressions that can be written down. One role that representations play is to enable us to break down proofs into manageable parts and thus to focus on certain details of a proof, by removing irrelevant information. [Carter 2010, 3]

Here I wish to show just how representations enable us to break a proof down, and explain in more detail the role that *icons* and *indices* play in this process.

My claim is simply that one role of iconic representations, i.e., representations that resemble what they represent, is to ensure that there is a link between the obtained result and the original expression. Use of indices makes it possible to reinsert the result obtained, by dealing with the manageable parts, in the appropriate place. A different aspect that will be addressed concerns what could be denoted as 'compound definitions'. By those, I mean definitions consisting of more than one component. In the case considered, we are dealing with matrices with certain entries. So one component of the definition tells us we are dealing with matrices, the second component concerns the entries of the matrices. The point here is that at a particular step of the proof one need only to focus on one component—thus removing irrelevant information.²

This process will be illustrated by an example from current mathematical practice, more precisely a result from Free Probability Theory. What is found is the value of an expression. By doing this my approach differs from previous work in this field in two respects. First, most examples in the literature are historical examples. Second, as many authors are concerned with visualisation, many examples concern geometry—and geometric diagrams. The example presented here concerns an algebraic expression.

Before turning to the case study, I will give a brief introduction to what will be needed from Peirce's theory of signs. Note my point is not to present—or argue for—a reading that Peirce would necessarily have endorsed. I present

2. Ken Manders [Manders 1999] deals more extensively with a similar point. He introduces the notions of responsiveness and indifference in order to address the topic of progress in mathematics. Human beings have limited resources for paying attention. As a consequence, one measure for the progress of a method is in terms of the number of types of response rendered superfluous by it.

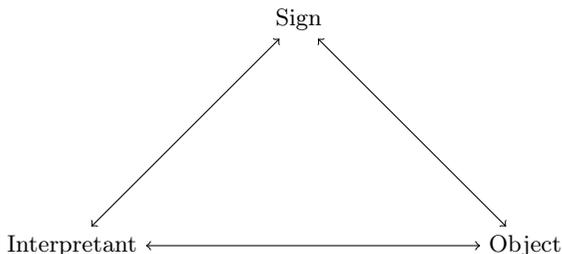
an interpretation of his theory that I believe is useful when addressing certain aspects of mathematical reasoning.³

1 Semiotics

The fundamental concept is that of a sign. A sign—or a representation—is something that stands for something else. According to Peirce, for something to act as a sign more is required; a sign is not a sign unless it is interpreted as a sign. A sign, therefore, is a triad (as everything else is understood in terms of three):

A sign, or *representamen*, is something which stands to somebody for something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates I call the *interpretant* of the first sign. The sign stands for something, its *object*. It stands for that object, not in all respects, but in reference to a sort of idea, which I have sometimes called the *ground* of the representamen. [Peirce 1965, 2.228, italics in original]⁴

What is notable in this description is that, in addition to the sign and the object that it represents, there is a third part, the interpretant. Thus a sign can be pictured as follows:



3. For more interpretations of Peirce's philosophy of mathematics I recommend the recent book edited by Matthew Moore [Moore 2010].

4. References to Peirce are to his Collected Papers, 1965. 2.228 means paragraph 228 in the second book.

Given this triad one may ask, for example, about the nature of the sign in question and about relations between sign and object.⁵ What is relevant for our purpose is the relation between the sign and the object; one may ask: In what capacity does the sign represent a given entity? According to Peirce's categories there are three possibilities, named icon, index and symbol. For a first description, icons represent by virtue of likeness. A picture is an example. An index represents by being actually connected to what it represents. The movement of the leaves and branches of the tree outside my window is a sign to me that it will be windy later when I cycle home. Finally a symbol represents by the assignment of a rule that determines its interpretant. Words are symbols, as well as, for example, π .

Peirce makes further subdivisions. Icons may in turn represent in three ways, depending on whether the sign shares certain qualities with the represented; or it shows some actual relations; or by virtue of "parallelism in something else" [Peirce 1965, 2.227]. These different kinds are denoted images, diagrams and metaphors.

Indices represent by being connected to the represented, either by a real physical—causal—connection, or by a purposeful act of connecting the signs. As an example of an index in the first sense, Peirce refers to a weather cock that is taken as a sign of the direction of the wind. For the second sense, Peirce mentions the geometers' use of letters, in the sense of signposts, in diagrams as indices.

Symbols obviously play a major role in mathematics as most expressions are built up from symbols. This is also the case with the expression that we will consider later. Here we will not be much concerned with the role that these play. I mention one curiosity:⁶ usually the symbol π is taken to denote the real number that denotes the relation between the circumference and the diameter of a circle. In the case study we shall deal with, π denotes a permutation. The rationale for this is that Greek letters are often used to denote mathematical entities and it is 'p' for permutation. The letters that we shall consider are the outcome of a "purposeful act of connecting a sign to its object", and they will therefore act as indices:

Geometricians mark letters against the different parts of their diagrams and then use these letters to indicate those parts. [Peirce 1965, 2.285]

In this sense, Peirce comments that "indices are absolutely indispensable in mathematics" [Peirce 1965, 3.305].

In addition Peirce explicitly highlighted the role of icons—"likenesses"—in mathematics:

5. According to Peirce's Phaneroscopy, sign, interpretant and object can take part in firstness (possibility), secondness (existence), and thirdness (law), which—at first—gives rise to 10 different categories of signs.

6. This point was made by one of the reviewers.

The reasoning of mathematicians will be found to turn chiefly upon the use of likenesses, which are the very hinges of the gates of their science. The utility of likenesses to mathematicians consists in their suggesting in a very precise way, new aspects of supposed states of things... [Peirce 1965, 2.281]

In the case study we shall deal with, there will be two senses of iconicity at play.

Central to Peirce's conception of reasoning in mathematics is that all such reasoning is diagrammatic—and as such iconic. A mathematical theorem will contain certain hypotheses. By fixing reference with certain indices, it is possible to produce a diagram that displays the relations of these referents. In statements concerning basic geometry, the diagram could be a geometric diagram as we know them from Euclid. In other parts of mathematics diagrams take a different form. One of Peirce's favourite examples from set theory concerns the theorem that—in modern terms—the cardinality of a set is strictly less than the cardinality of its power set. Denoting the set by C , it is possible to express the relations in a diagram as follows: *There is no 1-1 correspondence between the members of C and $P(C)$* . This is a statement about the relation between a set, C and the set of subsets of C , i.e., a diagram. It is thus clear that Peirce employs the term 'diagram' in a much wider sense than usual. Even spoken language can be diagrammatic:

the wordy and loose deductions of the philosophers may make use rather of auditory diagrams, if I may be allowed the expression, than [of] visual ones. [Peirce 1976, 164]

Generally a proof proceeds by—either seeing the conclusion of the theorem directly in the diagram—or by performing experiments on the diagram which will in the end lead to the conclusion. In the above example, the crucial step in the experiment consists in first assuming that the theorem is false, saying that there is a 1-1 correspondence $f : C \rightarrow P(C)$ and then forming the set $M = \{s \in C : f(s) \notin C\}$.

In our case study we see examples of diagrammatic reasoning in this sense. In addition, there will also be examples of reasoning using iconicity in the sense of a parallelism in something else, i.e. as a metaphor:

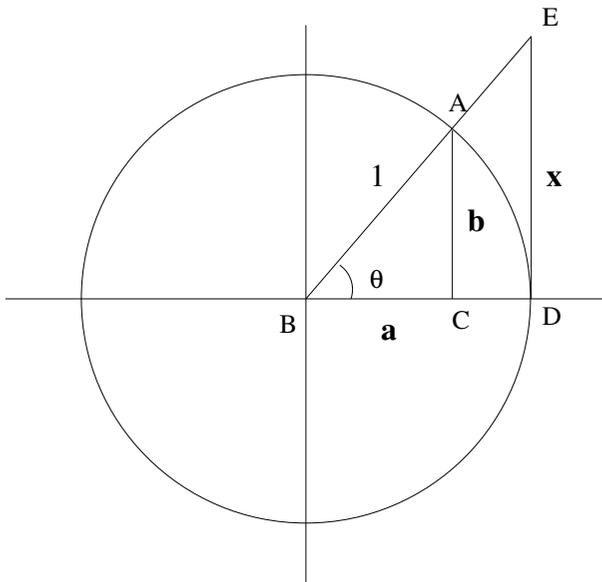
Particularly deserving of notice are icons in which the likeness is aided by conventional rules. Thus, an algebraic formula is an icon, rendered such by the rules of commutation, association, and distribution of the symbols. [Peirce 1965, 2.279]

To illustrate this consider the following identity: $(a + b)^2 = a^2 + 2ab + b^2$. To see that this holds, we use (conventional) rules that hold for multiplication and sum of numbers, and what it means for something to be "squared", i.e., to be multiplied by itself. We also need to know that $ab = ba$. Most notably,

we need to know, or be acquainted with, what the letters a and b are taken to represent, so they function as indices. If they represent numbers, as in this case, the commutative rule holds, if something else, this rule need not hold. Since—with the aid of these conventional rules—we obtain new insight, Peirce stresses that iconicity is at stake:

This capacity of revealing unexpected truth is precisely that wherein the utility of algebraical formulae consists, so that the iconic character is the prevailing one. [Peirce 1965, 2.279]

Before we turn to the main example, I shall illustrate some of the main ideas with a simpler example. The point is to simplify the expression $\sin(\tan^{-1}(x))$. The first step is to represent $\tan^{-1}(x)$ geometrically, see the picture below. To simplify, we assume $x > 0$.



Denote by θ the angle $\tan^{-1}(x)$. In this diagram, we can also identify $\sin(\tan^{-1}(x)) = \sin(\theta)$. To simplify, we denote this line segment a . From the diagram we see that we have two similar triangles, BAC and BED . Using the fact that the relation between corresponding sides of such triangles are identical, we can write the following equation:

$$\frac{a}{1} = \frac{b}{x}.$$

Since a and b are sides of the right angled triangle BAC , we can calculate that the following relation hold between a and b : $b = \sqrt{1 - a^2}$. We thus have the

following equation $a = \frac{\sqrt{1-a^2}}{x}$. Solving for a :

$$\begin{aligned} a &= \frac{\sqrt{1-a^2}}{x} \\ a \cdot x &= \sqrt{1-a^2} \\ a^2 \cdot x^2 &= 1-a^2 \\ a^2(x^2+1) &= 1 \\ a &= \frac{1}{\sqrt{x^2+1}} \end{aligned}$$

Summing up: In the first step we represent part of the expression in a geometric diagram, i.e., we display relations of the theorem in a diagram. It is also possible to identify the sought for expression, $\sin(\tan^{-1}(x))$, in this diagram as the length of a side in a certain triangle. By lettering these sides appropriately—using indices—we can express a relation between known x and unknown quantities a, b , and thus produce a new diagram. This diagram is an equation involving real numbers. Using rules for how to operate on these, we arrive at the final expression for a . Since a was the index representing the desired expression, we can reinsert this and obtain the result: $\sin(\tan^{-1}(x)) = \frac{1}{\sqrt{x^2+1}}$. Note the use of iconicity, both in the sense of producing certain diagrams, and as using conventional rules. In this example we also see that each step has a certain focus. When dealing with the first diagram, the focus is first on similar triangles. Next we focus on the right angled triangle. In the final step(s), we solve an equation involving real numbers. This is one aspect of what I refer to as “using representations in order to break a proof down into manageable parts”. Finally note the extensive use of indices. I explicitly mentioned this with respect to the ‘ a ’ that is used to denote the expression.

2 Case from Free Probability Theory

We will consider U. Haagerup’s and S. Thorbjørnsen’s paper [Haagerup & Thorbjørnsen 1999]. In this paper the following expression is studied

$$\mathbb{E} \circ Tr_n [(B^* \cdot B)^p],$$

where

- B denotes $m \times n$ “Gaussian Random Matrices”, thus in particular, they are matrices. Their entries are complex valued Gaussian random variables. B^* means the transpose of the matrix with conjugated entries.
- Tr_n denotes the trace of a matrix, and
- \mathbb{E} denotes expectation.

A Gaussian random matrix is a matrix whose entries are complex valued random variables of the form $f_{ij} + i \cdot g_{ij}$. The B_i 's are thus of the following form:

$$\begin{pmatrix} f_{11} + i \cdot g_{11} & \dots & f_{1n} + i \cdot g_{1n} \\ \vdots & & \vdots \\ f_{m1} + i \cdot g_{m1} & \dots & f_{mn} + i \cdot g_{mn} \end{pmatrix}$$

The entries, f_{ij}, g_{ij} , form families of independently but equally distributed Gaussian random variables.

This expression is first reduced to

$$\sum_{\pi \in S_p} \mathbb{E} \circ \text{Tr}_n [B_1^* B_{\pi(1)} \dots B_p^* B_{\pi(p)}],$$

where

- B_1, \dots, B_p are independent matrices of the same type as B .
- The π appearing in the subindices is a permutation on the set $\{1, 2, \dots, p\}$.

It is the following expression that will be the topic of our investigation:

$$\mathbb{E} \circ \text{Tr}_n [B_1^* B_{\pi(1)} \dots B_p^* B_{\pi(p)}] \quad (1)$$

In this particular case we take the mean of the entries of the matrices to be 0 and their variance 1, i.e., $f_{ij}, g_{ij} \sim N(0, 1)$. It is shown that the expression is equal to ($\hat{\pi}$ is a permutation obtained from π , $k(\hat{\pi})$ and $l(\hat{\pi})$ denote certain natural numbers, that will be explained below):

$$\mathbb{E} \circ \text{Tr}_n [B_1^* B_{\pi(1)} \dots B_p^* B_{\pi(p)}] = m^{k(\hat{\pi})} \cdot n^{l(\hat{\pi})}. \quad (2)$$

The proof is done in a number of steps that will be presented here, demonstrating how it is broken down into manageable parts. The focus is on the overall steps, keeping the mathematical details to a minimum. The proof can be construed as consisting of two major parts (as illustrated in picture 7). In one, the permutation π is reworked into a different permutation, $\hat{\pi}$. In the second, the focus is on the expression (1). We start with this expression.

1. In the first step the trace of the product of the matrices is found. In order to obtain this, the letters, B , are taken to represent ordinary matrices. We then use rules for multiplication of matrices and take the trace of the resulting matrix.⁷ Letting $b(u, v, k)$ designate the u, v 'th entry of the matrix B_k , fixing reference by indices, the following expression **(3)** is obtained:

7. We also need the fact that the expectation of the sum is the sum of expectations.

$$\sum_{\substack{1 \leq u_1, u_3, \dots, u_{2p-1} \leq n \\ 1 \leq u_2, u_4, \dots, u_{2p} \leq m}} \mathbb{E}[\overline{b(u_2, u_1, 1)} b(u_2, u_3, \pi(1)) \cdots \overline{b(u_{2p}, u_{2p-1}, p)} b(u_{2p}, u_1, \pi(p))].$$

Note the indices of u_i are divided into even and uneven parts, and i runs from 1 to $2p$. Odd indices keep track of columns and even ones keep track of the rows of the matrices. Recall that the entries $b(u, v, k)$ are (complex valued) Gaussian distributed random variables. They are distributed so that the value of these products—if not zero—is 1. This means that we simply can count how many of these terms in the sum are not zero. We may make an initial guess about the value of the expression simply by looking at the number of possible choices of the u_i 's. If i is odd, u_i can take all values between 1 and n . If i is even, u_i can take any value between 1 and m . Adding up, since there are p each of them, there are $n^p \cdot m^p$ different products. To find the value of the expression, we need to work out how many of these products are not zero.

2. Next the entries $b(u, v, k)$ are taken to designate complex valued random variables. The condition that the expectation of such a product is different from zero is that the product must consist of pairs of entries with their conjugate.⁸ This gives the following condition:

$$b(u_{2i}, u_{2i+1}, \pi(i)) = \overline{b(u_{2\pi(i)}, u_{2\pi(i)-1}, \pi(i))}, \quad i \in \{1, 2, \dots, p\}$$

or

$$u_{2i} = u_{2\pi(i)} \quad \text{and} \quad u_{2i+1} = u_{2\pi(i)-1}.$$

3. By calculations on these two conditions and use of the definition of $\hat{\pi}$ one obtains $u_j = u_{\hat{\pi}(j)+1}$. This equality tells us that when a product is non-zero, then this equality holds. The question is then how many groups of different u_j for $j \in \{1, 2, \dots, 2p\}$ there are. Mathematically this corresponds to determining the number of equivalence classes under the equivalence relation $j \sim_{\hat{\pi}} \hat{\pi}(j) + 1$ on the set $\{1, 2, \dots, 2p\}$.

Note that there is also a notational shift where $j \in \{1, 2, \dots, 2p\}$ replaces $2i + 1$ and $2i$.

It is seen that, through these steps, there is a gradual reduction of the original expression (1). The first expression concerns the expectation of a product of Gaussian Random variables. In the reduced expression $u_j = u_{\hat{\pi}(j)+1} - u_j$ as well as the indices j represent *natural numbers*. More precisely they represent variables that take as values certain natural numbers. (If j is odd u_j can take values from 1 to n , if even u_j can take values from 1 to m . j can take values from 1 to $2p$.) π in the indices represents a permutation. The

8. This follows since the expression $\mathbb{E}(\{f + ig\}^m \{f - ig\}^n)$ is zero unless $m = n$.

task of finding the value of the expression is finally reduced to considering an identity concerning natural numbers. This is one of the simpler details referred to in the above description of the role of representations in proofs. Note also that each step has a certain focus. In step one, the focus is on properties of matrices, forgetting that their entries are random variables. In the second step, the focus shifts to the entries, the complex valued random variables, and as already noticed, in the third step, we deal mainly with natural numbers.

Turning to the role of icons, we see that, at first, part of the theorem (2) is displayed in the expression (1). In step 1 above we use iconicity in the sense of a parallelism in something else. We take the indices, B_i , to denote matrices and use rules that tell us how to multiply these in order to obtain the product of the random variables.

At step 2 iconicity is used in the sense of producing a (new) diagram that depicts certain relations that will tell us something about the solution of the problem. Looking at expression (3), we see that it is a sum. The setup is such that each summand—if not zero—is one. So we need to count the number of terms that are not zero. The desired relations are obtained by asking for the condition—what are relations between random variables—so that product is not zero. The formulation of this condition becomes the new diagram. The rest of the proof works out these conditions using an interplay between these two roles of icons, until the basic step concerning natural numbers is arrived at.

The second part of the proof concerns the permutation $\pi : \{1, 2, \dots, p\} \rightarrow \{1, 2, \dots, p\}$ which is transformed into a new permutation $\hat{\pi}$ on $\{1, 2, \dots, 2p\}$. This also reflects the change of sub-indices in the expression above. One way to arrive at the expression for $\hat{\pi}$ is to write the product of the matrices without the permutation:

$$C_1^* C_2 \cdots C_{2p-1}^* C_{2p}$$

But we still wish to keep track of which matrices are identical, so we compare the above expression with the original one:

$$B_1^* B_{\pi(1)} \cdots B_p^* B_{\pi(p)}.$$

Before illustrating the general idea we consider an example, letting $p = 4$ and considering the permutation $\pi = (12)(34)$ ⁹:

$$C_1^* C_2 C_3^* C_4 C_5^* C_6 C_7^* C_8,$$

is compared to

9. $\pi = (12)(34)$ means that it is the map $\pi : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4\}$, where the number 1 is mapped to 2, 2 to 1, and 3 is mapped to 4, and 4 to 3.

$$B_1^* B_2 B_2^* B_1 B_3^* B_4 B_4^* B_3.$$

We see that the first and fourth B have the same index 1, which means they are identical, similarly the matrices in the second and third place are identical, continuing until we have:

$$C_1 = C_4 \quad C_2 = C_4 \quad C_5 = C_8 \quad C_6 = C_7$$

This particular example is shown in picture 3 below.

In general comparing the two expressions one obtains the following identities:

$$C_{2i-1} = B_i = B_{\pi(\pi^{-1}(i))} = C_{2\pi^{-1}(i)}$$

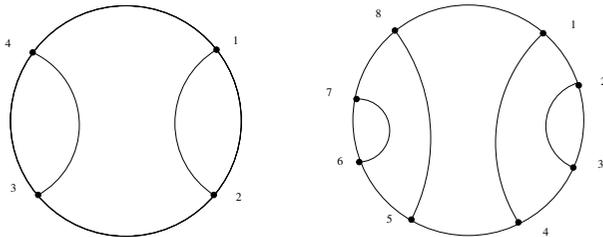
and

$$C_{2i} = B_{\pi(i)} = C_{2\pi(i)-1}.$$

Calculations on these equations give the permutation $\hat{\pi}$:

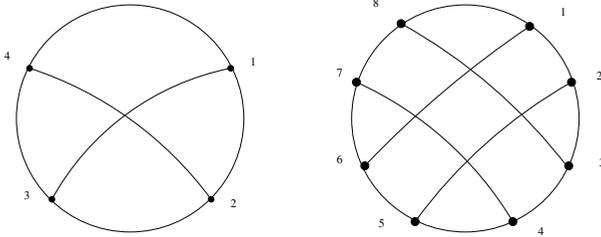
$$\begin{aligned} \hat{\pi}(2i - 1) &= 2\pi^{-1}(i), & i \in \{1, 2, \dots, p\} \\ \hat{\pi}(2i) &= 2\pi(i) - 1, & i \in \{1, 2, \dots, p\} \end{aligned} \tag{3}$$

Note that $\hat{\pi}$ takes odd numbers into even numbers and vice versa. These permutations can be pictured by diagrams. If $p = 4$, and π is the permutation (12)(34):



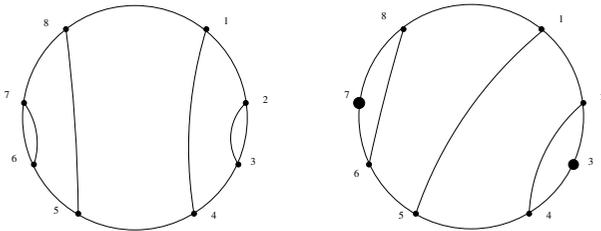
Picture 3. The diagram on the left represents $\pi = (12)(34)$ and the diagram on the right represents the corresponding $\hat{\pi}$.

For another example take the permutation (13)(24):



Picture 4. The diagram on the left represents $\pi = (13)(24)$ and the diagram on the right represents the corresponding $\hat{\pi}$.

Recall the equivalence relation obtained above. The number of equivalence classes can also be illustrated by diagrams. The relation is $j \sim_{\hat{\pi}} \hat{\pi}(j)+1$, and we use it on the two permutations above. The first gives the following equivalence classes:



Picture 5. The diagram on the left represents $\hat{\pi}$ from picture 3 and the diagram on the right represents equivalence classes. Numbers connected by a line are in the same equivalence class. These are calculated as follows:

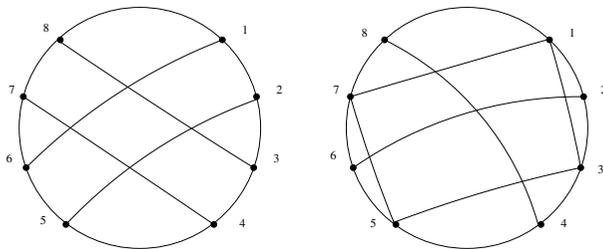
$$1 \sim_{\hat{\pi}} \hat{\pi}(1) + 1 = 5, \text{ so } 1 \text{ is seen to be related to } 5. \text{ Similarly}$$

$$5 \sim_{\hat{\pi}} \hat{\pi}(5) + 1 = 8 + 1 = 1, \text{ calculating mod } 8.$$

Thus 1 and 5 form an equivalence class.

In this case, there are 5 equivalence classes, of which 2 contain even numbers and 3 contain odd numbers. From the definition of the equivalence relation, it can be seen that equivalence classes will always contain exclusively even numbers or odd numbers. Thus we may define $k(\hat{\pi})$ to be the number of equivalence classes containing even numbers, and $l(\hat{\pi})$ to be the number of equivalence classes containing odd numbers. Then if the value (2) of the expression is recalled, it is seen that if p is 4, and the permutation is as just discussed, the value of the expression is $m^2 \cdot n^3$.

The second example gives:



Picture 6. *The diagram on the left represents $\hat{\pi}$ from picture 4 and the diagram on the right represent equivalence classes. Numbers connected by a line are in the same equivalence class. These are calculated as in the former example.*

Here are only 3 equivalence classes, which give the value $m^2 \cdot n$ to the expression. Since in the two cases there is a different number of equivalence classes, one may ask whether there is a way to determine how many there are for a particular permutation. This question is examined by the authors. It turns out that if the lines do not cross, as in the first example, there will always be 5 equivalence classes when $p = 4$. In general there will be $p + 1$ equivalence classes. This is another detail of the proof that can be studied partly using diagrams as addressed elsewhere [Carter 2010]. Here I will mention a single point concerning iconicity of the diagrams. The property of being a crossing or non crossing permutation has bearings on the result of the expression. This property is at first visible in the diagrams. It is possible, though, to formulate an algebraic analogue of being a crossing permutation. In this sense, since the diagram displays this relation, it is an iconic representation.

Moving on to explain the role of icons and indices, consider first the diagram below representing the process described above:

$$\mathbb{E} \circ Tr_n [B_1^* B_{\pi(1)} \cdots B_p^* B_{\pi(p)}] =$$

$$\sum \mathbb{E} [\overline{b(u_2, u_1, 1)} b(u_2, u_3, \pi(1)) \cdots \overline{b(u_{2p}, u_{2p-1}, p)} b(u_{2p}, u_1, \pi(p))]$$

$$= m^{k(\hat{\pi})} \cdot n^{l(\hat{\pi})}$$

$$u_j = u_{\hat{\pi}(j)+1}$$

$$j \sim_{\hat{\pi}} \hat{\pi}(j) + 1$$

$\pi \rightarrow \hat{\pi}$
 $u_{2i} = u_{2\pi(i)}$ and $u_{2i+1} = u_{2\pi(i)-1}$

Picture 7

We see again that the expression is finally, via a reformulation of the permutation, reworked to the equality $u_j = u_{\hat{\pi}(j)+1}$. The answer is then found from the equivalence relation $j \sim_{\hat{\pi}} \hat{\pi}(j) + 1$, calculating the number of equivalence classes. We have thus seen how the original expression is gradually broken down to study specific details.

The claims about the role of icons and indices were:

1. Icons, i.e., representations that resemble what they represent, enable the use of an obtained result in the original expression.
2. The role of indices is to make it possible to reinsert the result in the appropriate place.

The major letters used are the following:

- B , in the first step representing matrices, in the second step representing Gaussian random matrices.
- b , representing random variables.
- π , representing a permutation.
- u_j , representing natural numbers.

In the first step of the proof, we take the B_i 's to represent matrices and it is our acquaintance with these that give us rules on how to multiply, and find the trace of, matrices. These rules are used to obtain the second expression, the sum of expectations of the product of the entries. Observe that to obtain the first equality, we need both acquaintance with what the letters, B_i , are taken to represent together with relations that hold for these kind of entities. This was the description given above for iconicity in terms of a mediated third. The same picture holds in step three, where calculations are performed on the conditions

$$u_{2i} = u_{2\pi(i)} \quad \text{and} \quad u_{2i+1} = u_{2\pi(i)-1},$$

leading to the expression $u_j = u_{\hat{\pi}(j)+1}$. Here we take u_j to represent numbers, and use relations on these to obtain the desired equality. (This is simplified somewhat in order not to drown the main point in mathematical details.)

The second step is different. Here we note that the $b(u, v, k)$'s represent random variables, and that the expression involves products of complex random variables with their conjugates. From this expression, we *extract* certain relations that must hold in order for the products to be different from zero. These relations allow us to produce a new diagram.

Even though there is partial likeness along the way, there is a puzzle that needs to be assembled in the end. That this is possible depends on the indexical use of the representations, i.e., the role of representations as “sign-posts”, which is illustrated in the diagram.

- $\hat{\pi}$ points to π in the original expression.
- $u_j = u_{\hat{\pi}(j)+1}$ points—if the change of subindices is recalled—to the entries $b(u_{2i}, u_{2i-1}, k)$.

In conclusion, my claim was that the value of a complex expression could be found—using representations—in a series of steps to break it down to manageable parts. In each step it is possible to select a particular focus, and thus “remove irrelevant information”. Use of iconicity along the way, ensures that the final expressions or results tell us something about the original expression, whereas the use of symbols as indices lets us recall where to reinsert the obtained results.

Bibliography

CARTER, JESSICA

- 2010 Diagrams and proofs in analysis, *International Studies in the Philosophy of Science*, 24(1), 1–14.

DUVAL, RAYMOND

- 2006 A cognitive analysis of problems of comprehension in a learning of mathematics, *Educational Studies*, 61, 103–131.

GROSHOLZ, EMILY R.

- 2007 *Representation and Productive Ambiguity in Mathematics and the Sciences*, New York: Oxford University Press.

HAAGERUP, UFFE & THORBJØRNSEN, STEEN

- 1999 Random matrices and K-theory for exact C^* -algebras, *Documenta Mathematica*, 4, 341–450.

MANDERS, KENNETH

- 1999 Euclid or Descartes: Representation and responsiveness, unpublished.
- 2008 The Euclidean diagram, in *The Philosophy of Mathematical Practice*, edited by MANCOSU, PAOLO, Oxford: Oxford University Press.

MOORE, MATTHEW E.

- 2010 *New Essays on Peirce's Mathematical Philosophy*, Chicago and La Salle: Open Court Publishing Company.

PEIRCE, CHARLES SANDERS

- 1965 *Collected Papers*, Cambridge: The Belknap Press of Harvard University Press.
- 1976 *The New Elements of Mathematics*, vol. IV, The Hague: Mouton.