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The Right Order of Concepts: Graßmann, Peano, Gödel and the Inheritance of Leibniz's Universal Characteristic

Paola Cantù

Aix-Marseille Université
CNRS – CEPERC UMR 7304 (France)

Résumé : L'article aborde la question suivante : est-ce que le bon ordre des concepts peut être considéré un élément essentiel de rigueur scientifique dans la logique et les mathématiques de XIX^e et XX^e siècle, en particulier quand il s'agit d'auteurs qui ont été influencés profondément par le projet leibnizien de la caractéristique? L'article prend en considération trois exemples : Hermann Graßmann, Giuseppe Peano et Kurt Gödel. Selon notre thèse, le choix des concepts primitifs dans les théories hypothético-déductives n'était pas seulement une question d'opportunité, mais parfois aussi le résultat d'une investigation philosophique sur les fondements des disciplines scientifiques. La question du « bon » ordre des concepts n'est plus considérée comme une tâche réalisable, mais elle est devenue un idéal à suivre; néanmoins elle reste une partie essentielle du travail axiomatique. L'article vise donc à critiquer l'opposition trop nette qu'on trouve dans la littérature entre l'investigation de l'âge classique sur le bon ordre des concepts et la création de l'axiomatique moderne. La rupture scientifique déterminée par la création des systèmes axiomatico-deductifs en mathématiques et logique doit donc être associée à certains éléments de continuité qui regardent l'idéal de la connaissance en tant que recherche d'une théorie générale des concepts à obtenir par composition de certains éléments fondamentaux.

Abstract: This paper tackles the question of whether the order of concepts was still a relevant aspect of scientific rigour in the 19th and 20th centuries, especially in the case of authors who were deeply influenced by the Leibnizian project of a universal characteristic. Three case studies will be taken into account : Hermann Graßmann, Giuseppe Peano and Kurt Gödel. The main claim will be that the choice of primitive concepts was not only a question of convenience in modern hypothetico-deductive investigations, but sometimes

also the result of philosophical investigations onto the foundation of scientific disciplines. The question of the “right” order of concepts is an ideal to be followed rather than a task that can be fulfilled, but remains nonetheless an essential part of the axiomatic enterprise. This paper aims to question whether there is in fact such a stark contrast, as there is often claimed to be in the literature, between the debates relating to the right order of concepts and the foundational questions concerning modern axiomatics. The scientific rupture determined by the appearance of hypothetico-deductive systems in mathematics and logic should thus not be dissociated from some relevant continuities concerning the ideal of knowledge as the search for a general theory of concepts deriving from some fundamental elements.

1 Introduction

The question of the right order of concepts has traditionally been associated with the problem of rigour in mathematics. Aristotle's distinction between *ordo essendi* and *ordo cognoscendi* and the idea that what is first for us is usually not first in itself suggested that the search for rigour in science should include an analysis of the differences between the *ordo essendi* and the *ordo cognoscendi* and some kind of activity that could lead us from what is first for us to what is first in itself: dialectics plays this role in Aristotle's system [de Jong 2010].

The problem of the right order of concepts was particularly evident in mathematics and gave rise to criticism and proposals of revision of Euclid's *Elements*, as in the case of Port Royal Logic and Pierre de la Ramée's writings. The search for the right order of concepts could not be separated from the search for the right definitions and the fundamental concepts (considered either as first in themselves or as unanalysable, or as first for us).¹ It is traditionally believed that the distinction between *ordo essendi* and *ordo cognoscendi* got lost in hypothetico-deductive axiomatics, given that the primitive notions assumed in the axioms need not be concepts that are first in themselves at the ontological level nor concepts that are first for us at the epistemological

1. De Jong and Betti have tried to recall those aspects of the theory of knowledge in a scheme that they called *The classical model of science* and which attempts to describe the conception of scientific knowledge as a *cognitio ex principiis* [Betti & de Jong 2010].

This paper was actually first presented at the International Conference *The Classical Model of Science II. The Axiomatic Method, the Order of Concepts and the Hierarchy of Sciences from Leibniz to Tarski* organized at the Vrije Universiteit Amsterdam, August 2-5, 2011. What interested me in the model was the emphasis on the distinction between the *ordo essendi* (conditions 1-5 of the model) and the *ordo cognoscendi* (conditions 6-7 of the model).

level, but just concepts that can be more convenient or more fruitful. In particular, a contrast is often introduced between two ways of investigating the foundations of logic and mathematics and the role of axioms: the search for an exposition of scientific truths according to the “right” order of concepts and the presentation of scientific truths in hypothetico-deductive systems.

This paper claims on the contrary that there is more in common between the two approaches than one might expect, and in particular that the search for the “right” order of concepts plays a relevant role in the works of three authors who contributed greatly to the rise of modern axiomatics, and to the investigation into the foundations of mathematics and logic from the mid-19th century to the mid-20th century: Hermann Graßmann, Giuseppe Peano and Kurt Gödel. The choice of these authors is not arbitrary: all three authors were deeply influenced by Leibniz’s ideal of a universal characteristic, and shared a deep interest in the problem of primitive concepts and propositions. Given that Leibniz admitted the distinction between an *ordo essendi* and an *ordo cognoscendi*—as he distinguished for example synthesis from analysis²—it seems natural to raise the question of whether authors who explicitly related their own work to Leibniz’s characteristic might not have shared his idea that there is a right order of concepts.³ The paper isn’t aimed at pointing to a historical development from Graßmann to Peano and from Peano to Gödel,⁴—but rather at highlighting differences in their respective use of Leibniz’s ideas either to develop new approaches or to corroborate new results by means of an appeal to Leibniz’s authority. So the paper does not aim to historically investigate whether the mentioned authors actually inherited something from

2. “Synthesis is achieved when we begin from principles and run through truths in good order, thus discovering certain progressions and setting up tables, or sometimes general formulas, in which the answers to emerging questions can later be discovered. Analysis goes back to the principles in order to solve the given problems only, just as if neither we nor others had discovered anything before” [Leibniz 1683-1685, 232].

3. “As a boy I learned logic, and having already developed the habit of digging more deeply into the reasons for what I was taught, I raised the following question with my teachers. Seeing that there are categories for the simple terms by which concepts are ordered, why should there not also be categories for complex terms, by which truths may be ordered? I was then unaware that geometricians do this very thing when they demonstrate and order propositions according to their dependence upon each other” [Leibniz 1683-1685, 229]. See also [de Jong 2010, 239].

4. I have discussed this issue in other papers. See in particular [Cantù 2003, 332–337], where I discuss some differences between Graßmann’s calculus in the two editions of the *Extension Theory* and Peano’s axiomatization in *The Geometrical Calculus*: the latter is limited to the case of n dimensions and is not focused on the generation of the system. See also [Cantù forthcoming], where I compare several passages from Gödel’s unpublished philosophical manuscripts, the *Max Phil*, with relevant passages from Peano’s *Formulary* and from Russell’s *Principia Mathematica* on definite descriptions, definitions and functions, suggesting how intensively Gödel had worked on Peano’s writings and opposing, or at least restricting, the conceptual continuity between Peano and Russell outlined in recent literature.

Leibniz or not,⁵ but rather inquires as to what use they made of that inheritance. The main aim of the paper is to show the inadequacy of a historical framework that tends to eliminate from modern logic relevant philosophical issues, such as the problem of the right order of concepts—judging them to be merely superfluous or outdated questions.

It is true that recent trends in the history and philosophy of mathematics, especially the approaches based on mathematical practice, have rehabilitated several philosophical issues that had long remained unnoticed, such as the question of the purity of method, the explanatory power of proofs, or the relevance of mathematical values, either pragmatic or aesthetic. Yet, the role of the right order of concepts has not been fully investigated in this respect either. I cannot discuss this issue here—it could be the topic of a further separate paper—but I believe that when one has shown how much importance the problem of the right order of concepts had in the works of Graßmann, Peano and Gödel, then it would be quite natural to ask the question of whether there still is a role for this issue in the contemporary debate on the foundations of mathematics.⁶

This paper will investigate what Graßmann, Peano and Gödel thought about the realisation or realisability of Leibniz's project of a universal characteristic, and in particular whether they conceived the characteristic as one or many, and how they considered primitive concepts occurring in it.⁷ The

5. Concerning the relation of Graßmann to Leibniz, the debate—which I briefly reconstructed in [Cantù 2003, 319–320]—involved among others [Couturat 1901], [Rothe 1916], [Heath 1917], [Lotze 1923], [Barone 1968], [Freudenthal 1972] and [Muenzenmayer 1979]. Recent literature generally agrees on the idea that Graßmann's project had not been directly influenced by Leibniz's perspective, but on different grounds. Echeverría claimed that there is a huge difference in generality between Leibniz's *analysis situs* and Graßmann's geometrical calculus [Echeverría 1979], i.e., a different level of generality, which gets lost in Graßmann, because he introduces equality instead of congruence, whereas Otte [Otte 1989] remarked that the difference concerns the abandonment of the ontological foundation of classical epistemology. De Risi [De Risi 2007, 111–112] recalls Graßmann's opportunism, because he clearly adapted his previous work for the 1846 Jablonowski Prize, but mentions also some aspects where Graßmann's perspective is truly Leibnizian. I agree with the idea that Graßmann had not been directly influenced by Leibniz's writings, but I also claim that the effort to present his own work in relation to Leibniz's project had some effects on his philosophical approach (see further section 3).

6. My intuition would be that the epistemological question can be dealt with from the perspective of the inquiries into mathematical values and mathematical styles, rather than on the basis of investigations into the kind of mathematical rigour granted by axiomatics. Yet, the question would be whether some metaphysical traits of the question might fail to be adequately analysed from this perspective, and might require an interdisciplinary approach that takes into account the relations between scientific, philosophical and theological domains.

7. The emphasis on these issues was suggested to me by the interpretation presented by Francesco Barone in the introduction to an Italian edition of Leibniz's "logical" writings which, although largely unknown, presents several reasons of interest [Barone 1968].

focus on these authors and these questions will allow us to show not only that Leibniz left behind an important legacy in that period (from the mid-19th to the mid-20th century) when the modern axiomatic was developed—a result that has already been largely investigated in the literature⁸—, but also that the use of Leibniz’s ideas was considerably different for each of the mentioned authors. What will be specifically investigated in this paper is the role assigned to the search for a “right” (i.e., ontologically or epistemologically prior) order of concepts: was it totally abandoned or did it survive in modified forms? The focus on this question will guide the brief exposition of certain aspects of Leibniz’s project of a universal characteristic in the next section and the choice of some specific quotations that will be useful for a textual comparison with remarks by Graßmann, Peano and Gödel in the next sections.

2 The heritage of Leibniz’s characteristic

2.1 Leibniz’s characteristic and the right order of concepts

Leibniz’s characteristic is based on the search for a small number of *primitive* notions that might be identified by some fundamental characters, so that the complex notions could be obtained and their features described from the combination of the fundamental characters.⁹ As Couturat remarked, Leibniz’s idea is based on the belief that for each simple concept there might be a symbol that expresses it in the most *natural* way, so that all complex concepts might be expressed naturally as combinations of the former.¹⁰

The question of the right order of concepts is here related to the question of the distinction between simple and complex ideas. It is not mainly related to the problem of guaranteeing a sure foundation of scientific knowledge, but rather to the possibility of building a synoptic table where each idea finds its own place, and solutions to problems can be found more easily.¹¹ The improvement in heuristic efficiency provided by the characteristic is compared to the improvement in sight guaranteed by technological instruments such as

8. See for example [Heinekamp 1986], [Krömer & Chin-Drian 2012], and especially [De Risi 2007] for the history and the success of the *analysis situs*.

9. “As a matter of fact, when thinking about these matters a long time ago, it was already clear to me that all human thoughts may be resolved into very few primitive notions; and that, if characters are assigned to them, it will then be possible to form characters for the derived notions, from which it will always be possible to extract all their conditions, as well as the primitive notions they contain, and—let me say explicitly—their definitions or values, and therefore, the properties, which may be deduced from the definitions as well” [Leibniz 1684, vol. 7, 223, Engl. transl. 182].

10. “Leibniz’s characteristic is the search for the right and natural symbols to express an idea as decomposed in its *fundamental* parts” [Couturat 1901, 76].

11. See [Leibniz 1683-1685, 232], quoted above in footnote 2.

the microscope, or the telescope, but it is considered to be more powerful, because it does not limit itself to an improvement of a sensorial faculty but rather it improves the power of reason.¹²

An essential trait of Leibniz's characteristic is its general applicability to forms, i.e., not only to quantities but also to qualities. More precisely, it is the possibility of applying the characteristic to order, similitude, and relation that enables its application to quantities and not vice versa [Leibniz 1695, 61].¹³ The characteristic concerns *geometry*, and might be applied to *physics*. Algebra describes only quantitative aspects of things; a new geometrical analysis is needed to express position: its characters could thus represent figures, but also machines and movements.¹⁴

Here Leibniz seems to consider algebra and the geometrical calculus as two parallel treatments of quantities and figures respectively. Yet in other passages he considers algebra itself as an application of the combinatorial art considered as a general theory of abstract forms that concerns metaphysics, thereby basing the question of the right order of concepts on metaphysical grounds [Leibniz 1715, 24, Engl. transl. 669].

2.2 A tension in Leibniz's idea of a characteristic

There is a peculiar tension in what Leibniz says about the relationship between the characteristic applied to specific domains and a general characteristic.¹⁵ Is a philosophical analysis of the first metaphysical principles and fundamental

12. "Once the characteristic numbers of most notions are formed, humankind will have a new type of instrument which will enlarge the mind's power to a far greater degree than the eyes' power was increased by optical lenses, an instrument as superior to microscopes and telescopes as reason is superior to sight. No magnetic needle ever offered greater comfort to seamen than this Little Dipper (*cynosura*) shall offer those traversing a sea of experiences" [Leibniz 1679c, 268], Engl. transl. in [Leibniz 2008, 124].

13. See also the following passage: "This art is distinct from common algebra, which deals with formulas applied to quantity only or to equality and inequality. This algebra is thus subordinate to the art of combinations and constantly uses its rules. But these rules of combination are far more general and find application not only in algebra but in the art of deciphering, in various games, in geometry itself when it is treated linearly in the manner of the ancients, and finally, in all matters involving relations of similarity" [Leibniz 1683-1685, 233].

14. "But in spite of the progress which I have made in these matters, I am still not satisfied with algebra, because it does not give the shortest methods or the most beautiful constructions in geometry. This is why I believe that, so far as geometry is concerned, we still need another analysis which is distinctly geometrical or linear and which will express *situation* [*situs*] directly as algebra expresses *magnitude* directly. And I believe that I have found the way and that we can represent figures and even machines and movements by characters, as algebra represents numbers or magnitudes" [Leibniz 1679b, 568-569, Engl. transl. 248-249].

15. This point was clearly made by Francesco Barone [Barone 1968, lxix-lxxi]. A similar distinction has been recently introduced by O. Pombo [Pombo 1988], who

ideas necessary to develop the characteristic¹⁶ or is the latter independent from true philosophy?¹⁷ In other words, is the characteristic a metaphysical instrument that should determine the absolutely primary concepts or a way to constitute specific scientific domains? For example, is binary arithmetic only useful to determine the properties of natural numbers or is it essential to describe the metaphysical fact that all things derive from God and nothing?¹⁸ This tension is clearly reflected in the problem of whether the concepts that are assumed as primitive in a science are merely first for us or first in some objective sense. This would imply a conception of axiomatics that does not restrict itself to the *ordo cognoscendi*, but should ideally converge with an axiomatics based on the *ordo essendi*.

What holds for the characteristic language holds for the characteristic calculus too: there is a tension between specific calculi, like the geometric calculus, and the idea of a general calculus—calculus ratiocinator—that should operate on real characters.¹⁹ So, on the one hand Leibniz considers that there might be as many characteristics as there are domains of investigation, while on the other hand he aims to develop a characteristic of all characteristics. On the one hand he claims that the characteristic might be developed independently from philosophy (e.g., in specific domains such as arithmetic, or geometry) and on the other hand he suggests that it is subordinated to the development of true philosophy, because only the latter can tell which concepts are really fundamental.

These two issues are strictly combined in Leibniz's writings and the tension remains unresolved. The question of the right order of concepts applies both in the case of specific characteristics and in the case of a general characteristic, but its answer is clearly different depending on the relation with true philosophy. In the next two sections we will consider whether this tension, that is so typical of Leibniz, remained in the works of successive authors such as Graßmann, Peano and Gödel, and in particular whether they considered the order of concepts as a matter of scientific rigour, and whether philosophy played a role in it.

claimed that the ambiguity is to be found in Leibniz's own texts and is not only a matter of interpretation. Donald Rutherford [Rutherford 1998], although accepting Pombo's suggestion, rather presents the distinction as pertaining to two different interpretations, preferring the standard analytical reading promoted by Russell [Russell 1900]: the merit of Leibniz is certainly related to his idea of a formal symbolism, where the characters can be seen as devoid of contents and their deductive relations are made explicit.

16. This is what Leibniz claimed in a letter to Burnet dated 24 August 1697 [Leibniz 1875-1890, vol. 3, 216].

17. See Leibniz's unpublished remark [Leibniz 1966, 27-28].

18. This example is based on a passage from *De organo sive arte magna cogitandi* [Leibniz 1698, 239].

19. This is another point made by Francesco Barone [Barone 1968, lxxii-lxxiii].

2.3 Three different heritages: Graßmann, Peano and Gödel

Graßmann, Peano, and Gödel all presented themselves as inheritors of Leibniz's tradition. Yet they developed different aspects of Leibniz's philosophical project, thereby defending different conceptions of mathematical rigour and a different understanding of the role of philosophy in the search for primitive concepts and propositions. There might be opposite explanations of this fact: either they did not really take inspiration from Leibniz, but just made recourse to his authority as a precursor in order to legitimize their innovations; or they inherited only one of several ideas that were already in tension in Leibniz's thought. My claim is that the truth is somehow in-between these two interpretations. All three authors became truly interested in Leibniz's logical work, and made frequent references to Leibniz's project of a characteristic, because they were fascinated by it; yet, they developed their original mathematical and logical results quite independently from Leibniz's results. What they shared was an interest in Leibniz's philosophical project and in the possibility of accomplishing it. Yet, they had different epistemological and philosophical perspectives, which guided them to different readings of Leibniz, all somehow faithful to the texts.

The positions concerning the role of rigour in mathematics were very different, as follows: a) Hermann Graßmann was particularly keen on linguistic rigour, introducing a new technical terminology for newly introduced concepts; b) Giuseppe Peano considered rigour as a preliminary and essential condition of any mathematical discourse (without rigour, one does poetry, but not mathematics [Peano 1891, 66]): rigour is associated with the lack of contradiction and with the possibility of giving a proof; and c) Kurt Gödel's idea of rigour was related to the clarification and analysis of concepts that might make them sufficiently precise.²⁰ The same notion of rigour, based on the idea of the clarification of concepts, applies to philosophy: for this reason, Gödel seems to be convinced of the possibility of developing philosophy into a rigorous science.²¹

The relations between mathematical and philosophical foundation were also different: a) According to Graßmann, philosophy played an essential role in the determination of the primitive concepts of mathematics: the division of mathematics itself into four branches and a general theory that precedes them

20. "In the second case it must be possible, after making the concepts in question precise, to give a rigorous proof for the existence of that necessity" [Gödel 1972, 274].

21. "I am under the impression that after sufficient clarification of the concepts in question it will be possible to conduct these discussions with mathematical rigour and that the result then will be that (under certain assumptions which can hardly be denied [in particular the assumption that there exists at all something like mathematical knowledge]) the Platonistic view is the only one tenable" [Gödel 1951, 322–323].

is grounded on a philosophical deduction; b) Peano always tried to separate the foundational role of philosophy from the foundational role of mathematics: the former discusses the origin of concepts and the latter the choice of the smallest number of primitive concepts and propositions that allow the derivation of all truths; and c) Gödel wondered whether the search for the right primitives in mathematics and logic should not depend on the search for the right primitives in philosophy and metaphysics, thereby associating the problem caused by paradoxes in set theory with a lack of rigour (i.e., the lack of a sufficiently precise clarification of concepts) in philosophy and metaphysics.

In this paper I will claim that the differences in their understanding of mathematical rigour and in the conception of the relations between mathematics and philosophy can be understood as different ways of inheriting some aspects of Leibniz's idea of a characteristic, and in particular the idea of a right order of concepts. Differences between the authors might be explained by the fact that 1) some of them believed that a characteristic of all characteristics should be developed while others believed that there should be as many characteristics as there are domains of investigation; 2) some of them claimed that the characteristic should be developed independently from philosophy (e.g., in specific domains such as arithmetic, or geometry), whereas others claimed that it should be subordinated to true philosophy, which determines the fundamental concepts.

3 Graßmann

The tension between the strive towards a characteristic of all characteristics and the construction of different characteristics based on different domains of investigation cannot be found as such in the texts by Hermann Graßmann: his writings were aimed at distinguishing the specific geometric calculus from the more general characteristic.²² It is well known that Hermann Graßmann reacted explicitly to Leibniz's idea of a characteristic expressed in the letter to Huygens, first published in 1833, in his essay *Geometrische Analyse* written for the 1846 Jablonowski Prize, which asked to develop the Leibnizian idea of a geometric characteristic, and to build a calculus that might express Leibniz's ideas [Graßmann 1847]. I will not here enter into the discussion as to whether Graßmann's aim was the same as Leibniz's:²³ for the scope of the present paper it is sufficient to remark that Graßmann declared that he had developed his work independently but tried to present it as a realization of Leibniz's

22. This is in particular the point where my interpretation differs from that of Echeverría [Echeverría 1979]. Graßmann was fascinated by Leibniz's strive for generality, but was interested in the construction of a specific calculus: the geometric one.

23. See footnote 5.

project.²⁴ Apart from the contingent application to the prize, which made it necessary to show that his own theory was somehow related to Leibniz's project, it is interesting to remark that Graßmann's defense of his new theory was based on the great number of its possible applications. As a matter of fact, extension theory included an abstract foundation of vectorial spaces and a treatment of extensive multiplicities with n dimensions that could be applied to the specific case of 3-dimensional geometry. Besides, the new calculus could be applied to the whole of physics²⁵ and to non spatial objects too, because the calculus might become independent from spatial intuition,²⁶ as Leibniz himself had claimed [Leibniz 1679b, 571].

The previous remarks suggest that Graßmann meant to develop a specific calculus and not a general characteristic, but a calculus that might have a variety of different applications. Not only did he avoid any mention of the role of philosophy in the search for the relevant elements of this calculus, but he aimed at separating the geometrical calculus from the general characteristic.²⁷ So, Graßmann could be considered as the one who first separated (long before Russell) the project of a universal language made of real characters from the project of building a symbolical calculus.

24. "In order to let the scientific meaning of [Leibniz's] peculiar characteristic come into light also otherwise, and in order to make the scientific gain in this domain more intuitive from another point of view, I will take the following line in the derivation and development of the new analysis. I will assume the Leibniz's characteristic, and show how the analysis that I am inclined to see as a realisation, even if only a partial one, of Leibniz's idea of a geometrical analysis emerges from this nucleus—by implementation and further development, by an appropriate elimination of what is extraneous and by fertilization with the ideas of geometrical affinity. That this is not the path along which I have arrived at this analysis does not even need to be mentioned here" [Graßmann 1847, 327–28].

25. "So, I think I have shown in the application to mechanics introduced above how mechanics can be effectively treated in a pure geometrical way by means of this analysis [...] I could have easily given other examples from optics, acoustics, electrodynamics and other branches of physics" [Graßmann 1847, 397–398].

26. "Finally there is at the end of Leibniz's presentation still a peculiar place where he clearly enunciated the applicability of this analysis also to objects that are not of spatial nature [...]. And one can easily see, once one has accepted this idea of a pure conceptually grasped passage, that also the laws developed in this section are capable of being conceived independently from spatial intuition. In this way Leibniz's thought is realized" [Graßmann 1847, 398].

27. "Leibniz himself definitely distinguished his thoughts about a pure geometric analysis, whose development and achievement floated before his eyes as a far objective, even if he fully recognised its importance, from his search for a new characteristic, which he connected to the former in order to make the possibility of the realisation of those thoughts more believable and to leave a monument to posterity, in case he should be hindered from its achievement. The two need to be sharply separated, if one wants to rightly appreciate the merit of Leibniz in the development of the geometrical analysis" [Graßmann 1847, 326]. Cf. also the passage quoted in footnote 24 on page 166, where Graßmann remarks that Leibniz's geometrical calculus needs to be separated from what is extraneous to it.

Echeverría is right as he remarks upon the decrease in generality in Graßmann's geometrical calculus, but he does not consider that the idea of a general characteristic can be found elsewhere in Graßmann's works, even if Graßmann does not explicitly associate it to Leibniz nor to the idea of a characteristic of characteristics [Echeverría 1979]. What can be found in Hermann Graßmann's mathematical treatises and in the works of his brother Robert Graßmann is the concept of a unified analysis of concepts. The *General Theory of Forms* developed by Hermann is a preliminary investigation of the fundamental thought operations that occur in any mathematical domain (logic, arithmetic, geometry, combinatorics, extension theory) [Graßmann 1844, 33ff.], while the *Theory of Forms* developed by Robert is a science based on qualitative besides quantitative relations that should generally be valid for all human beings, whatever their nation or their language [Graßmann 1872, § 1, 6].

So, the tension between the idea of one characteristic and the development of applied or specific characteristics has not completely vanished in Graßmann's works, although, as Michael Otte rightly observed [Otte 1989], the metaphysical and ontological foundation has been abandoned. Yet, the question of the right order of concepts is still present: on the one hand it emerges in the philosophical deduction by means of which Graßmann introduces a partition of the general theory of forms into four independent but parallel branches; on the other hand the choice of the primitive concepts and the order of the proofs is not at all arbitrary in *Extension Theory*.

For example, one of the differences between Peano's calculus and Graßmann's theory concerns the choice of the notion of dimension as more primitive than the concept of base: although theorems and proofs can be compared, the philosophical idea behind Graßmann's project is lost, if one changes the order of concepts. Peano first takes a system of entities of a certain dimension as given and then introduces a way to obtain it from a subset of its elements that are linearly independent. Graßmann on the contrary first takes the operation that determines a set of (independent) generators as primitive, and then considers the systems it can give rise to. The right order of concepts appears thus related not only to the degree of generality or efficiency in proofs,²⁸ but also and most importantly to the assumption of the primacy (*for us* and thus also *per se*, given Graßmann's idealism) of operations on their products.²⁹

Hermann Graßmann did not introduce an axiomatic theory of extensive quantities or of arithmetic in the same sense as Peano or Dedekind, but he developed an analysis of the primitive concepts of both sciences and also of the

28. For a more detailed comparison of these aspects in Peano and Graßmann, see [Cantù 2003, 332ff.].

29. For an analysis of the role played by the operation of multiplication in Graßmann's mathematical theory and its epistemological and philosophical import concerning the difference between numbers and magnitudes, the relation between geometry and extension theory, and the development of a constructivist approach to mathematics, see [Cantù 2010, 98–100].

general theory of forms. In the case of mathematics the primitive concepts are considered to be fundamental, because they are obtained by a philosophical deduction, i.e., by a dichotomic division that is similar to the Platonic procedure by *diaeresis*. Yet, unlike the Platonic dichotomy, Graßmann's philosophical deduction proceeds by intersection of the opposites and not only by successive divisions, and is not followed by a movement that goes back from the multiplicity of construed concepts to the unity of the starting point. The starting point of the deduction (i.e., the division of sciences into formal and real, of generating acts into continuous and discrete, and of elements into equal and different) is not verified, and the dialectic division is justified by the correspondence with acts of thought [Cantù 2003, 172]. Mathematical primitives are the couples of opposites equal / different and continuous / discrete; the logical primitives, presented in the *allgemeine Formenlehre* as general forms, are: equality, difference, connection [*Verknüpfung*] and separation [*Sonderung*].

So, the primitive concepts occurring in the general theory of forms and in mathematics are the same and depend on philosophy: on the one hand, because they are justified by a philosophical deduction; on the other hand because they correspond to the fundamental acts of thought. The order of primitive concepts cannot be arbitrarily changed, but rather is strictly related to the correspondence between subjective and objective levels that is typical of idealism.

4 Peano

While Hermann Graßmann seemed mainly interested in a specific geometrical calculus (*analysis situs*), Giuseppe Peano explicitly described Leibniz's project of a *Speciosa Generalis* as a sort of universal Language or Writing System, where the symbols guide reasoning.³⁰ Quoting Leibniz's essay on the universal characteristic, Peano recalls that this discovery is taken to be more important than telescopes or microscopes: it is the polar star of reasoning.³¹ Besides, Peano shared Leibniz's aim to determine a very small number of primitives, and his concern for the identification of symbols that could naturally express ideas and their reciprocal relations. Yet, in the *Formulary* one finds only a specific symbolic system and calculus concerning logical and mathematical truths. Here, the symbols guide reason inasmuch as different symbols denote

30. "Gottfried Wilhelm Leibniz during all his life (1646-1716) was concerned with a kind of 'Speciosa generalis' where all truths of reason are reduced to a sort of calculus. At the same time it could be a sort of universal language or writing system, but infinitely different from those that have been planned until now, because the characters and even the words would thereby direct reason" (*Opera philosophica* a. 1840, 701) [Leibniz 1679a], [Peano 1896, 1].

31. "He considers this discovery as more important than the invention of telescopes and microscopes; it is the North Star of reasoning" [Peano 1896, 1].

different ideas, whereas the same symbol is used when the difference between two words is grammatical rather than conceptual.³²

Unlike Graßmann, Peano believes that the construction of a symbolic system should not be limited to mathematics. Yet, given that this enterprise goes beyond the possibility not only of a man but of a whole research group, needing the effort of the whole of society,³³ Peano and his collaborators contented themselves with the application of the symbolic notation to the analysis of mathematics.³⁴ So, according to Peano it is perfectly possible to develop a specific calculus without having to preliminarily establish a general characteristic: the advantage (both foundational and didactical) of each part is already evident before the whole is completed.³⁵ The relative independence of specific calculi from a complete analysis of ordinary language—which needs nonetheless to be accomplished, because it is useful to distinguish ambiguities and avoid imprecise formulations—also depends on the provisional nature attributed to the *Formulary*: even if it is a collection of truths and not of conventions, it can be corrected and ameliorated by the comments and criticism of contributors, like a collaborative Dictionary or a Wiki that can be implemented by its own readers.³⁶ Besides, Peano was particularly impressed by the aim of Leibniz's project: the solution of verbal controversies and the search

32. "The study of different properties of ideas represented by the symbols ϵ and $\supset$ prevents us from representing them by the same symbol, even if they correspond in language to more or less the same word 'to be'. The identity of the expressions 'it is contained' and 'one deduces' shows us that there is only a grammatical difference between them and leads us to denote them by the same symbol $\supset$. And so on. Changing the forms of the signs ϵ , $\supset$, ... does not change those truths" [Peano 1896, 1].

33. "This project is undoubtedly beautiful. Unfortunately its execution goes beyond the energy, not only of a man, but of several men. Only a numerous and well organized society could accomplish it" [Peano 1896, 4].

34. "We have already applied those results both to enunciate certain propositions precisely, and to analyse some complete theories, especially relative to the still controversial principles of mathematics" [Peano 1896, 3].

35. "Because it is not necessary that all this work be achieved in order to be fruitful. Each published part is already useful to students of those particular subjects" [Peano 1896, 4].

36. "Does one want to study a topic whatsoever? One just needs to open the *Formulary* at the right page, because it is possible to order the topics according to the signs that compose them, just as one orders words in a dictionary according to the letters that constitute them. In a few pages one will find all known truths on that topic, together with their proofs and historical information. Should the reader know any proposition that he might have discovered or found in some book, or should he notice any inaccuracy in those propositions, he might convey those additions and those corrections to the Editorial Board of the *Formulary*: they will be announced in some periodical publication and will be taken into account for the next edition" [Peano 1896, 2].

for a unique notation. This is particularly evident in Peano's own remarks³⁷ and in the remarks made by other members of the school. Giovanni Vailati for example considered one of the merits of Peano's enterprise and more generally of logical pragmatism that of identifying different historical theories (known under different names) as having the same content, thus avoiding sterile oppositions.³⁸ Burali-Forti and Marcolongo remarked upon the importance of the introduction of a non-arbitrary, unique notation in order to improve the diffusion of new theories such as the vector calculus.³⁹

According to Peano, a symbol is primitive with respect to a given set of symbols if it is not defined by means of those symbols. Being a primitive is a relative and not an absolute property of symbols. Primitive symbols denote ideas that are considered as primitive in a given axiomatic system: e.g. the system of natural, rational, real numbers, etc. Properties of primitive symbols are expressed in the primitive propositions (axioms), which might serve as definitions of the primitive terms. Not all ideas expressed by the primitive symbols of a system need be fundamental ideas, as already proved by Peano's remarks on the redundancy of the logical symbols introduced in the *Formulary*.⁴⁰

Peano's symbols do not express exactly the same concept in all contexts: the symbol of equality is defined for example as a relation of equivalence between individuals in one section and as mutual implication between propositions in another section. If the logical symbols, although used in mathematical sections, change their meaning according to the context, they express different concepts in different sections of the *Formulary*: therefore they cannot be taken to express a list of fundamental concepts that ground all knowledge. This is also due to the fact that Peano always has a privileged model for his axiomatic systems and introduces local definitions for the symbols.

37. See for example the numerous quotations given in [Luciano 2012], who claims that Peano's interest for Leibniz was mainly guided by the search for a precursor of his own work.

38. "A third point of contact between pragmatists and mathematical logicians consists in the interest shown on both sides for historical research into the development of scientific theories. [...] To this tendency to recognise the identity of theories, beyond or under differences of expression, symbolism, language, representative conventions and the rest, is to be attributed also the constant interest of the mathematical logicians in linguistic questions—from Graßmann, at once the author of the *Ausdehnungslehre* and of the *Wörterbuch zum Rig-Veda*, to Nagy, student of the transmission of Greek thought through the Syriac and Arabic commentaries; from Couturat, joint author with Leau of a history of the projects of 'Universal Language', to Peano, inventor and propagandist of one of the most practical among them: the 'latino non flexo'" [Vailati 1906, 691–692].

39. "The reason why this admirable means of research and presentation [the vector calculus] spread slowly and is still accepted suspiciously, is the fact that different authors use different names and signs to indicate the same vectorial entities" [Burali-Forti & Marcolongo 1907-1908, I, 324].

40. Peano's logical primitives terms are: ϵ , $\supset$, $=$, $\cup$, $\cap$, $-$, $\wedge$. Some terms in this list are redundant but useful for reasons of clarity and simplification of the derivations.

Terms might belong to specific parts of mathematics (geometry, arithmetic) or be common to all of them. Mathematical logic studies relations and operations that occur with the same properties in different branches of mathematics, and that should thus be expressed by the same symbol. Primitive terms are not fundamental: the choice of the terms used to denote the fundamental concepts might vary according to didactical needs and several alternative definitions of the primitive terms are possible. Besides, philosophy does not play any significant role in the determination of the primitive concepts.

Apparently, there is no interest in the question of the right order of concepts in the *Formulary* and in Peano's understanding of axiomatics. Yet, Peano's choice of symbolism reveals an effort to mirror the concepts by means of the symbols used to denote them. Peano, like Leibniz, insisted on a natural relation between the symbol and what it designates: this is clear in his choice of the symbol for "being a member of"—a Greek epsilon that stands for *est*—, or of the inverse *iota*, which expresses the inverse of the operation expressed by the *iota*. Leibniz's idea of a characteristic containing "real" characters is not completely abandoned in Peano's perspective. It emerges with even more force in Peano's investigations into a universal language, because the *latino sine flexione* should be based on symbols (roots of Latin words) that should preserve the essential relation to the denoted concept, independently from grammatical variations.

5 Gödel

Kurt Gödel considered Frege's and Peano's mathematical logic as a realisation of Leibniz's project of a general characteristic.⁴¹ Yet, he clearly remarked that mathematical logic was but a part of Leibniz's project, even if a central part of it, given that it is a science that is prior to all others and contains the principles underlying all sciences. In particular, he saw in the unaccomplished philosophical analysis of the primitive concepts occurring in the axioms the reason for the partial success of such calculi.⁴² The unsatisfactory analysis of the primitive concepts is responsible, according to Gödel, for the paradoxes of

41. "On the other hand, [mathematics] is a science prior to all others, which contains the ideas and principles underlying all sciences. It was in this second sense that mathematical logic was first conceived by Leibniz in his *characteristica universalis*, of which it would have formed a central part. But it was almost two centuries after his death before his idea of a logical calculus really sufficient for the kind of reasoning occurring in the exact sciences was put into effect (in some form at least, if not the one Leibniz had in mind) by Frege and Peano" [Gödel 1944, 119].

42. "Many symptoms show only too clearly, however, that the primitive concepts need further elucidation. It seems reasonable to suspect that it is this incomplete understanding of the foundations which is responsible for the fact that mathematical logic has up to now remained so far behind the high expectations of Peano and others who (in accordance with Leibniz's claims) had hoped that it would facilitate theoretical mathematics to the same extent as the decimal system of numbers has

set theory. Even if Russell's simple theory of types and axiomatic set theory can avoid all known paradoxes,⁴³ Gödel seems to be unsatisfied with restriction of types and with the extensional interpretation of sets for other reasons.⁴⁴

Such reasons are philosophical and are related to Gödel's interpretation of Leibniz's *characteristica* as a non-utopian project,⁴⁵ based on the idea that "everything in the world has a meaning", and aiming at clarifying concepts so as to develop an intensional logical theory. The clarification of concepts is important not only from the point of view of the *ordo essendi*, but also from the point of view of the *ordo cognoscendi*, because a correct analysis of mathematical concepts might immediately lead to the solution of mathematical problems.⁴⁶ How can the clarification of concepts take place? Gödel adopted the same metaphor used by Leibniz and recalled by Peano: primitive concepts have to be discerned like stars in the sky. In his conversation with Wang Gödel remarked that Leibniz had assumed seven primitive concepts in analogy with the Great Bear constellation.⁴⁷ Gödel suggested that a potentiation of

facilitated numerical computations. For how can one expect to solve mathematical problems systematically by mere analysis of the concepts occurring if our analysis so far does not even suffice to set up the axioms?" [Gödel 1944, 140].

43. "Major among the attempts in this direction (some of which have been quoted in this essay) are the simple theory of types (which is the system of the first edition of *Principia* in an appropriate interpretation) and axiomatic set theory, both of which have been successful at least to this extent, that they permit the derivation of modern mathematics and at the same time avoid all known paradoxes" [Gödel 1944, 140].

44. For a survey of Gödel's readings of Leibniz, see [Crocco 2012], who presents—in opposition to [Parsons 1990]—a detailed interpretation of the 1944 paper as focused on several Leibnizian issues that Russell had failed to solve adequately, in the belief that a good solution might only come from a return to logic as the science of all sciences. For a critical remark on the effectiveness of the analogy with monadology used by Gödel in order to justify the reflection principle in set theory, see [van Atten 2009].

45. "But there is no need to give up hope. Leibniz did not, in his writings about the *characteristica universalis*, speak of a utopian project" [Gödel 1944, 140].

46. "The epistemological problem is to set the primitive concepts of our thinking right. For example, even if the concept of set becomes clear, even after satisfactory axioms of infinity are found, there would remain more technical (i.e., mathematical) questions of deciding the continuum hypothesis from the axioms. This is because epistemology and science (in particular, mathematics) are far apart at present. It need not necessarily remain so. True science in the Leibnizian sense would overcome this apartness. In other words, there may be another way of analyzing concepts (e.g., like Hegel's) so that true analysis will lead to the solution of the problem" [Wang 1996, 237].

47. "The fundamental principles are concerned with what the primitive concepts are and also their relationship. The axiomatic method goes step by step. We continue to discover new axioms; the process never finishes. Leibniz used formal analogy: in analogy with the seven stars in the Great Bear constellation, there are seven concepts. One should extend the analogy to cover the fact that by using the telescope we [now] see more stars in the constellation' [Wang 1996, 297]. Actually Leibniz used the term *cynosura* (see above p. 162), which might mean either the constellation containing the Polar Star, i.e., the Ursa Minor or Little Bear constellation, or the Polar star itself,

sight could lead to the individuation of other primitive concepts: symbolism can be used to potentiate our capacity of distinguishing concepts just as the telescope is used to discern more stars.⁴⁸ The search for primitive concepts should not concern only mathematical logic, but should be extended to all concepts: Leibniz's characteristic is understood by Gödel as a general science. The following analogy from the *Philosophical Manuscripts* confirms this interpretation: Leibniz's *scientia generalis* is to scientific phenomena (sciences) as Newtonian physics is to physical phenomena. Leibniz's *scientia generalis* is interpreted by Gödel as a characteristic of all characteristics which introduces a constellation of concepts that apply to all phenomena—i.e. to all sciences, whereas Newtonian physics introduces a constellation of concepts (point of the space, point of time, point of mass, position, force, mass) that apply only to physical phenomena.⁴⁹

The clarification of concepts allowed by the general characteristic will grant a rigorous discussion of the foundations of mathematics⁵⁰ and a mathemat-

as interpreted by Peano (see footnote 31 on page 168). Gödel's confusion might have arisen from the fact that both constellations contain seven stars.

48. "The undefined concepts are those that are so bright (clear) that it is enough to say: look approximately in this or that direction (of the sky). In the other concepts the word is constructed by means of definitions. The feeling that only mathematical concepts and propositions are precise derives from the fact that those concepts are the most simple (bright), and therefore they are the first to be seen precisely" [Die undefinierten Begriffe sind die, welche so hell (deutlich) sind, dass es genügt zu sagen: Schaue ungefähr in diese oder jene Richtung des Himmels. Bei den anderen wird das Wort erst durch Definition konstruiert. Das Gefühl, dass nur die mathematischen Begriffe und Sätze präzise sind, kommt daher, dass diese Begriffe die einfachsten (hellsten) sind und daher am ersten präzise gesehen werden] [Gödel forthcoming, IX, 89–90]. Passages from Gödel's *Max Phil* are quoted also in the German original, given that they have been recently transcribed from handwritten notes, and—being still unpublished—are not easily accessible to the reader.

49. "Leibniz's *scientia generalis* is clearly something similar with respect to the whole domain of phenomena, i.e., all sciences—including mathematics—as Newtonian physics is with respect to physical phenomena. The 'Cynosura notionum' consists there of point of the space, point of time, point of mass, position, force, mass. Projecting all physical phenomena onto this system, i.e., trying to use them to interpret phenomena, the possibilities that subsist a priori are limited and predictions become possible" [Die *scientia generalis* des Leibniz ist offenbar etwas Ähnliches hinsichtlich des ganzen Gebiets der Erscheinung d.h. aller Wissenschaften, inklusive Mathematik wie die Newtonsche Physik hinsichtlich der physikalischen Erscheinungen. Die 'Cynosura notionum' besteht dort aus Raumpunkt, Zeitpunkt, Massepunkt, Lage auf, Kraft, Masse. Dadurch, dass man alle physikalischen Erscheinungen auf dieses System 'projiziert', d.h. es durch sie zu 'interpretieren' sucht, werden die a priori bestehenden Möglichkeiten eingeschränkt, und es sind daher Voraussagen möglich] [Gödel forthcoming, X, 67–68]. See also other passages from the *Philosophical Manuscripts*: [Gödel forthcoming, IX, 85; IX, 90 and X, 2–3].

50. "I am under the impression that after sufficient clarification of the concepts in question it will be possible to conduct these discussions with mathematical rigour and that the result then will be that (under certain assumptions which can hardly be de-

ically rigorous analysis of metaphysical and theological concepts.⁵¹ If the analysis correctly separates concepts that are mixed up at first sight, new fundamental concepts will be discovered and their analysis will lead to the solution of scientific problems, even if the procedure for solving problems cannot be a completely mechanical one: according to Gödel, Leibniz was wrong on this point.⁵²

Primitive concepts are, according to Gödel, the concepts we start out from, and also the concepts that cannot be derived from others.⁵³ Axioms are the primitive propositions of a theory and express the properties of primitive concepts. Primitive concepts have to be looked for not only in mathematical or scientific disciplines, but also in philosophy and in theology. The search for primitive concepts in logic does not amount to an axiomatisation of mathematical logic, but rather concerns the basic elements of a general theory of concepts.⁵⁴

Some primitive concepts might be fundamental in the sense that we must assume them as given in order to develop an axiomatic system. They appear as the most clear concepts that we have, as concepts that are primary according to the *ordo cognoscendi*.⁵⁵ But there is a second sense in which prim-

nied, in particular the assumption that there exists at all something like mathematical knowledge) the Platonistic view is the only one tenable" [Gödel 1951, 323].

51. "The famed philosopher and mathematician Leibniz attempted to do this as long as 250 years ago, and this is also what I tried to do in my last letter. The thing that I call the theological worldview is the concept that the world and everything in it has meaning and sense [*Sinn und Vernunft*], and in particular a good and unambiguous [*zweifellosten*] meaning. From this it follows directly that our presence on Earth, because it has of itself at most a very uncertain meaning, can only be the means to the end [*Mittel zum Zweck*] for another existence. The idea that everything in the world has a meaning is, by the way, exactly analogous to the principle that everything has a cause, which is the basis of the whole of science" [Wang 1996, 108].

52. "In 1678 Leibniz made a claim of the universal characteristic. In essence it does not exist: any systematic procedure for solving problems of all kinds must be nonmechanical" [Wang 1996, 202].

53. "Given any set of conceptions, in the sense of concepts with associated beliefs about them, we can try to determine what the reliable basic beliefs about each concept are; whether some of the concepts can be defined in terms of others; and whether some beliefs can be derived from others. Often we find that some concepts can be defined by other concepts, so that we can arrive at a subset of primitive concepts and construe all the beliefs in the set as concerned with them. Those beliefs in the initial set of beliefs which cannot be derived from other beliefs in the set are then taken as the axioms" [Wang 1996, 334–335].

54. "Gödel often speaks of an axiomatic theory or system in quite a loose way, so that he considers it necessary to find axioms for arithmetic, for geometry, for physics, but also for philosophy, and for theology. He also aims at finding the primitive concepts of logic as a general theory of concepts", see [Wang 1996, 334].

55. "The notion of existence is one of the primitive concepts with which we must begin as given. It is the clearest concept we have. Even 'all', as studied in predicate logic, is less clear, since we don't have an overview of the whole world. We are here thinking of the weakest and the broadest sense of existence. For example, things

itive concepts might be fundamental. Several passages from the *Philosophical Manuscripts* suggest that Gödel aimed to distinguish logic from psychological concepts, objective from subjective relations between concepts: like Frege, he claimed that the two levels are relevant in order to build a general theory of concepts. Primitive concepts might be fundamental both as psychological and as logical concepts, but in the first case this just means that we cannot do without them, in the second case this means that they are the most simple concepts that enter in the composition of all other concepts. The distinction between the subjective and the objective level allows the distinction between the epistemological order of concepts (*ordo cognoscendi*) and the “right” or “natural” order of concepts (*ordo essendi*).⁵⁶ Unlike Peano, Gödel attributed an important role to philosophy in the search for the primitive concepts of the general characteristic, but it is only in the interplay and the reciprocal influence between the particular sciences and philosophy that the task of finding the right order of concepts might be accomplished.⁵⁷

Even the analysis of logical primitives involves questions that can probably be answered only by the introduction of metaphysical questions.⁵⁸ Given the

which act are different from things which don't. They all have existence proper to them” [Wang 1996, 150].

56. “Even if we might not have access to it, there seems to be a right or natural order of primitive concepts and propositions that we should look for. The search for the right primitives and axioms is a philosophical task, based on the decomposition of concepts in simpler parts. The faculty that allows us to perceive concepts might be helped by instruments such as symbolism, just as our faculty of sight is improved by instruments such as the telescope (cf. Leibniz’s passage)” [Wang 1996, 234].

57. “My work with respect to philosophy should consist in an analysis of higher concepts (logical and psychological), i.e., what should be done is to write a list of those concepts and to consider the possible axioms, theorems and definitions for them (of course together with the application to the empirically given reality). But in order to do that, one should first obtain through (half understood) philosophical lectures, a ‘feeling’ of what one might assume. On the other hand, the understanding of an axiomatic would also increase the understanding of philosophical authors (so there is a reciprocal action from ‘top’ and ‘bottom’, whereby the correct behavior is important).” [Meine Arbeit in Bezug auf Phil<osophie> soll in einer Analyse der obersten Begriffe bestehen (der logischen und psychol<ogischen>); d. h. was letzten Endes zu tun ist, ist eine Liste dieser Begriffe aufschreiben und die möglichen Ax<iome>, Th<eoreme> und Def<initionen> für sie überlegen (selbstverständlich samt Anwendung auf die empirisch gegebene Wirklichkeit). Um das aber tun zu können, muss man zuerst durch (halb verstandene) phil<osophische> Lektüre ein ‘Gefühl’ dafür erwerben, was man annehmen kann. Andererseits wieder wird das Verstehen einer Axiomat<ik> das Verständnis der phil<osophischen> Schriftsteller erhöhen (also Wechselwirkung von ‘oben’ und von ‘unten’, wobei das richtige Verhältnis wichtig <ist>).] [Gödel forthcoming, IX, 78–79].

58. “Logical questions that are not mathematical and not psychological are those concerning logical primitive concepts, for example: belongs to, concept, proposition, class, $\supset$, relation. So, e.g.: if there is a concept for each propositional function, if there are classes that contain themselves, if all concepts are everywhere defined. These questions trespass into the domain of metaphysics and can probably be de-

idea that there are some fundamental concepts that philosophy should investigate and ultimately determine, even if this task has not been accomplished yet, Gödel's conception of axiomatics shows some affinity with the Classical Model of Science mentioned at the beginning. Like Bolzano, Gödel used the distinction between *ordo essendi* and *ordo cognoscendi* to explain why a list of fundamental primitives had not yet been given. Notwithstanding the inevitable discrepancy between the two levels, the search for rigour is based on the ideal convergence between *ordo essendi* and *ordo cognoscendi*: the ultimate task is to find the primitive concepts that are also fundamental at the objective level. Yet, this task can never be fully accomplished, because the determination of the primitives and of their correct relations and properties would amount to the solution of all problems, and thus to the elimination of human incompleteness, which is on the contrary an intrinsic and essential property of our existence as finite beings.

So, the task of determining a general characteristic is at the same time something we should strive towards and believe in—because there is no reason to give up hope—and a task that can never be fully accomplished, given our finite nature. It is an ideal that guides axiomatics but can never be fully reached.⁵⁹ It is thus not surprising that Gödel's determination of the primitives of a general theory of concepts was never definitively achieved. Yet, in his conversation with Wang and in the *Philosophical Manuscripts* he mentioned provisional lists of logical primitives,⁶⁰ and evaluated several metaphysical concepts in order to understand which could be considered as most fundamental.

As we saw in section 2, Leibniz's project of a characteristic was based on the idea that symbols should express concepts in a *natural* way. Is there any inheritance of this idea in Gödel? In the procedure known as Gödelization, Gödel used the first thirteen prime numbers to represent the most relevant logical terms. The choice of designating logical symbols by numbers is not only a matter of efficiency or fruitfulness. If one analyses some remarks that occur

cided only by the introduction of mere metaphysical concepts." [Logische Fragen, die einerseits nicht mathematisch und nicht psychologisch sind, sind solche, welche die logischen Grundbegriffe betreffen, z. B. ε , Begriff, Satz, Klasse, $\supset$, Relation. Also zum Beispiel: Gibt es zu jeder Aussagefunktion einen Begriff, gibt es Klassen, die sich selbst enthalten, ist jeder Begriff überall definiert. Diese Fragen greifen in das Gebiet der Metaphysik über und können wahrscheinlich nur mit Einführung rein metaphysischer Begriffe entschieden werden.] [Gödel forthcoming, IX, 62].

59. Analogously, Bolzano had claimed that there are some fundamental concepts from which all other concepts and propositions on them can be derived, although a list of these fundamental concepts cannot be given once and for all. See for example Bolzano's remarks on the concepts "having" and "quality": one is simple and one is composed from the other, but which one is simple cannot be determined with certainty [Bolzano 1837, §80, 184].

60. "Gödel mentioned the following list of logical primitives of a general theory of concepts: negation, conjunction, existence, universality, object, the concept of concept, which all belong to predicate logic, and the relation of application (which is specific to a theory of concepts)" [Wang 1996, 277].

in Gödel's *Philosophical Manuscripts*, it emerges that he was not insensible to the problem of an analogy between the signs and the things denoted by the signs, as in a passage where he discussed whether binary numbers could be more apt than decimal numbers to express the fundamental concepts.⁶¹ This remark about binary numbers is even more interesting if compared with other passages from the *MaxPhil*, where the number one is associated to God and to full existence.⁶² This means that Gödel's preference for the binary system is related to the capacity of its signs to express some fundamental metaphysical ideas. The right order of concepts depends thus on the choice of the right primitive metaphysical concepts.

6 Conclusion

Discussing the legacy of Leibniz's *characteristica* in the works of Graßmann, Peano and Gödel, this paper has shown that, apart from several differences, all three authors took the task suggested by Leibniz seriously, and tried to develop the idea of a general characteristic. Together with the project of the characteristic, they inherited some unresolved tensions that can be found in Leibniz's writings. Gödel and Graßmann, more than Peano, investigated the possible relations between a general and specific characteristics, and, unlike Peano, assigned a relevant role to philosophy in the search for primitive concepts and primitive propositions. A clear distinction between the *ordo essendi* and the *ordo cognoscendi* allowed Gödel to explain how the fact that there is a unique true order of concepts might be compatible with different orders developed by axiomatic systems. Although he believed that there might be some fundamental concepts, Peano did on the contrary consider the choice of the primitives and the order of concepts as something that might be changed ac-

61. "The designation of numbers in the dual system is more similar to a real 'ideography' (i.e., there are more properties that can be deployed from the symbols and there is less arbitrariness in the designation) than the decimal system. In the latter for example all numbers from 1 to 10 are designated in a fully arbitrary way, whereas in the dual system this is the case only for 0 and |, but one can prescind from this too when one considers the mere sequential structure. The less arbitrary designation is certainly ||||, and this apparently gives the most faithful 'image' of numbers." [Die Bezeichnung der Zahlen im Dualsystem kommt einer wirklichen 'Begriffsschrift' näher (d.h., es sind mehr Eigenschaften unmittelbar aus den Symbolen abzulösen, und es herrscht weniger Willkürlichkeit in der Bezeichnung) als die Dezimale. In dieser <ind> z.B. alle Zahlen von 1 bis 10 völlig willkürlich bezeichnet, in der dualen nur 0 und |, aber auch von dieser <ist> abzusehen, wenn man die bloße Reihenstruktur betrachtet. Am wenigsten willkürlich ist freilich die Bezeichnung ||||, und diese gibt scheinbar das treueste 'Bild' der Zahlen] [Gödel forthcoming, X, 80]. See also the following passage from the *Philosophical Manuscripts*: XI, 112–113. Note that this example is the same one as mentioned by Leibniz (see above p. 163).

62. "Only God exists, God is One" [Gott allein ist, Gott ist Eines] [Gödel forthcoming, IX, 51]

cording to didactical needs, and never mentioned the idea of a unique “natural” order of concepts.

The analysis of these three case-studies shows that the choice of primitive concepts was not only a question of convenience in modern hypothetico-deductive investigations, but sometimes also the result of philosophical investigations into the foundation of scientific disciplines. The question of the “right” order of concepts became an ideal to be followed rather than a task that can be fulfilled but remained nonetheless an essential part of the axiomatic enterprise. The scientific rupture determined by the appearance of hypothetico-deductive systems in mathematics and logic should thus not be dissociated from some relevant continuities concerning the ideal of knowledge as the search for a general theory of concepts deriving from some fundamental elements.

The notion of mathematical rigour inherited from Leibniz concerned the philosophical analysis of concepts as well as deduction. For this reason, it was not fully dissociated from the belief in an ideal, “natural” order of concepts that should orientate the search for the most fundamental primitives.

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