



Philosophia Scientiæ

Travaux d'histoire et de philosophie des sciences

18-3 | 2014

Logic and Philosophy of Science in Nancy (I)

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Electronic version

URL: <http://journals.openedition.org/philosophiascientiae/972>

DOI: 10.4000/philosophiascientiae.972

ISSN: 1775-4283

Publisher

Éditions Kimé

Printed version

Date of publication: 1 October 2014

Number of pages: 53-70

ISBN: 978-2-84174-689-7

ISSN: 1281-2463

Electronic reference

Werner Stelzner, « Nicolai Vasiliev's *Imaginary Logic* and Semantic Foundations for the Logic of Assent », *Philosophia Scientiæ* [Online], 18-3 | 2014, Online since 28 November 2014, connection on 03 November 2020. URL : <http://journals.openedition.org/philosophiascientiae/972> ; DOI : <https://doi.org/10.4000/philosophiascientiae.972>

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Nicolai Vasiliev's *Imaginary Logic* and Semantic Foundations for the Logic of Assent

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Résumé : Le philosophe russe Nicolai Vasiliev est connu en tant que précurseur des logiques essentiellement non-classiques, c'est-à-dire de logiques qui diffèrent de la logique classique par l'abandon de principes qui sont corrects en logique classique. La gamme de telles logiques couvre la logique intuitionniste, la logique plurivalente, la logique paraconsistante et les logiques de la pertinence. Dans la première partie de ce texte, j'analyse brièvement les vues de Vasiliev, à savoir sa « logique imaginaire », qu'il présente comme une nouvelle logique non-aristotélécienne. Dans les sections suivantes je discute l'impact des approches de Vasiliev sur la logique du consentement. Ici, les idées essentiellement non-classiques de Vasiliev seront reliées à des logiques non-essentiellement non-classiques, qui ne constituent pas une révision de la logique classique, mais étendent sa puissance expressive.

Abstract: The Russian philosopher Nicolai Vasiliev is known as a forerunner of substantially non-classical logics, i.e., logics that differ from classical logic by dropping principles that are sound in classical logic. The range of such logics covers intuitionistic logic, many-valued logic, paraconsistent logic and relevant logics. In the first part of this paper, I will give a short analysis of Vasiliev's views, namely his *Imaginary Logic*, which is presented by Vasiliev as a new non-Aristotelian logic. In the following parts I will discuss the impact of Vasiliev's approaches on the logic of assent. Here Vasiliev's essentially non-classical ideas will be connected with non-substantially non-classical logics, which don't constitute a revision of classical logic, but an expansion of the expressive power of classical logic.

The Russian philosopher Nicolai Vasiliev is known as a forerunner of substantially non-classical logics, i.e., logics that differ from classical logic by

dropping principles that are sound in classical logic. The range of such logics covers intuitionistic logic, many-valued logic, paraconsistent logic and relevant logics. However, Vasiliev's basic idea of non-Aristotelian logic is not strictly directed to the confinement of classical logic. He addresses a fundamentally changed logic, which not only excludes some principles of classical logic but can also include the adoption of logical principles that are absent from classical logic. Vasiliev intends to develop a genuine non-classical logic, "to demonstrate that a new logic and other logical operations than those which we use are possible, to show that our Aristotelian logic is only one among many possible logical systems. This new logic will not be a new presentation of the old logic; it will [...] be the "new logic" and not a new treatise concerning logic" [Vasiliev 1912, 53 f.].

As one example of this new non-Aristotelian logic Vasiliev gives an outline of the *Imaginary Logic*, that does not contain all the classically sound logical principles. However, the *Imaginary Logic* does not contain classically unsound principles. In this sense, Vasiliev's attempt to construct a non-Aristotelian logic is less radical than his basic remarks about the new logic.

If we take a closer look at Vasiliev's foundations for his *Imaginary Logic*, we can see that these are not convincing at each point. Furthermore Vasiliev has some illusions about the relation between his *Imaginary Logic* and classical Aristotelian logic.

1 Vasiliev's basic ideas about his *Imaginary Logic*

The main defect of Vasiliev's justification for the possibility of a non-Aristotelian logic can be seen in the ontological dimension of this justification. If treated epistemically, Vasiliev's approaches can acquire a clearer and more adequate foundation. Furthermore, if treated epistemically, the conception of *Imaginary Logic* can then find fertile applications for the development of epistemic logics, especially for the logic of assent.

Vasiliev started his argumentation for the possibility of a non-Aristotelian logic from an ontological point of view. He advanced the idea that logical laws mirror the features of the world and the soundness of logical laws is determined by the features of the world. Accordingly, in different worlds different logical laws can be sound:

While the Aristotelian logic is true for our world, the non-Aristotelian logic can be true only in some different world.
[Vasiliev 1912, 54]

However, this different world is not an alternative real world, but an imaginary world, a world purely ideally constructed:

The new logic lacks this connection with our reality; it is a purely ideal construction. Only in another world than ours, in an imaginary world (whose basic qualities we can exactly define, by the way) the imaginary logic can become a tool for producing knowledge. [Vasiliev 1912, 54]

The last quotation suggests that imaginary logic is only applicable if our world is replaced by an imaginary world. Such a replacement is not excluded logically. However, it is pure fiction. According to Vasiliev it is not possible that the two worlds, our real world and the imaginary world (as another kind of real world) can exist together. The logical laws of a world are only sound if this world is real. Because our world and the imaginary world cannot co-exist, the logical laws of our world (the laws of Aristotelian logic) and the logical laws of the imaginary world (the laws of imaginary logic) cannot both be sound together. The soundness of the formulae of classical and imaginary logic exclude each other. From this, Vasiliev draws the conclusion that a “contradictory” opposition exists between Aristotelian logic and imaginary logic, confusing “contradictory” with “contrary”:

The formulae of both logics will stand in a contradictory opposition: the truth of the formulae of imaginary logic excludes the truth of the formulae of our Aristotelian logic and vice versa. Because of this, not both formulae can be true for one and the same world. [Vasiliev 1912, 54]

Vasiliev was able to avoid drawing conclusions about the rival soundness of Aristotelian logic and imaginary logic by dropping his treatment of the imaginary world as an ontological world and adopting an epistemic interpretation of the imaginary world. However, there could then exist in our world (with the soundness of Aristotelian logic) different epistemic worlds. The soundness of the imaginary logic for these worlds and the soundness of Aristotelian logic for our world do not exclude each other.

If we take into account Vasiliev's concrete views about imaginary logic, we realize that his assertion about the contrary relation between imaginary and Aristotelian logic is not only misleading but clearly false. Imaginary logic is a partial logic derived from classical logic: all the laws of imaginary logic are sound laws of classical logic. However, the opposite is not true.

According to Vasiliev, his imaginary logic is constituted from classical Aristotelian logic analogously to the composition of non-Euclidean geometry from Euclidean geometry, where a sound principle of the latter is omitted:

The non-Aristotelian logic is the [Aristotelian – W.S.] logic without the law of the contradiction. Here it will not be redundant to add that just the non-Euclidean geometry served us as a model for the construction of the non-Aristotelian logic. [Vasiliev 1912, 54]

However, in imaginary logic there is no replacing law, opposite to the law of contradiction, which would stand in a contrary opposition to the law of contradiction. If we express the chief difference between the two logics, only meta-assertions are true that contradict each other. For the classical Aristotelian logic it holds that “The law of contradiction is true” and for imaginary logic that “The law of contradiction is not true”. But it is not generally true that a formula is a law of imaginary logic just because it is not a law of Aristotelian logic. And of course it is not true that there are laws contradicting each other in both Aristotelian and imaginary logic. The new (imaginary) logic contains only laws of the old (Aristotelian) logic. Vasiliev justifies the possibility of a new logic by indicating the possibility of composing new axiomatic systems from given systems by omitting one or more axioms from the given system:

One receives the logic just from the synthesis of some more independent Axioms (footnote: The mathematical logic can serve as an elegant proof of that, having several axioms and postulates as its basis) [...] We must come to the conclusion that rejecting some axioms and the construction of a logic without them is completely conceivable. [Vasiliev 1912, 57f.]

With his rejection of the law of contradiction in imaginary logic Vasiliev characterizes his non-Aristotelian logic as a kind of paraconsistent logics. Imaginary logic is a logic without the law of contradiction, but it is not a logic with the negation of the law of contradiction as a new law of this non-Aristotelian logic. Thus, Vasiliev does not maintain his initial view, that in non-Aristotelian logic laws are included that contradict the laws of Aristotelian logic. Consequently, Vasiliev does not accept the negation of the law of contradiction as a law in imaginary logic. In imaginary logic neither the law of contradiction $\neg(p \wedge \neg p)$ nor its negation $p \wedge \neg p$ are sound logical laws. So, being a supporter of paraconsistent views in logic, Vasiliev is not a supporter of a strictly dialectic view, where the negation of the law of contradiction would be true. Imaginary logic is a logic without the limitations connected to the law of contradiction:

The imaginary logic is a logic which is free of the law of the contradiction. [Vasiliev 1912, 59]

The soundness of the law of contradiction in Vasiliev’s view is bound to kinds of ontology, with the following characterization:

A cannot be non-A. No object includes a contradiction in itself. [Vasiliev 1912, 59]

We have this kind of ontology in our real world. Furthermore, because of this, there cannot be any doubt about the validity of the law of contradiction in our real world. Nevertheless, one can imagine other ontologies with other logical laws. With the determination of the soundness of logical laws, these laws are conceived by Vasiliev as empirical laws, which can differ according to

different features of worlds. Just for our real world Vasiliev states that there are situations, in which both an assertive judgment and its negation about the same object can be true. This is not determined by logic, but by the features of the world. This world can be otherwise, and then, e.g., the law of contradiction can lose its soundness. According to Vasiliev, if we wish to apply imaginary logic without the law of contradiction, we have to leave our world or our world should be changed. In our real world there is no place for imaginary logic.

However, if we treat imaginary logic in an epistemic perspective, there will be a place for the soundness and correct application of this logic without changing or leaving our real world. In our real world, epistemic situations (or epistemic worlds) are given, whereas epistemic subjects have contradictory epistemic attitudes or perform contradictory epistemic or linguistic acts. These epistemic attitudes or acts belong to our real world. However, the logical relation between them is not governed by Aristotelian logic with its rule of contradiction, but by epistemic logics without the law of contradiction. The existence of epistemic contradictions is entirely compatible with the soundness of the law of contradiction in our world. We don't have to leave our world, to obtain the possibility of a correct application of Vasiliev's imaginary logic. We merely have to refer to the possibility and existence of epistemic worlds, worlds of assent or imagination, in which both a sentence and its negation can be assented to by the same epistemic subject.

Of course, when Vasiliev emphasizes the unsoundness of the law of contradiction in imaginary logic (and in imaginary worlds), from the viewpoint of epistemic worlds and epistemic logic, only one of the main differences from classical Aristotelian logic is highlighted. Another important feature of these epistemic worlds, which distinguishes them from the worlds fitting for Aristotelian logic, is their incompleteness. In epistemic worlds we have sentences such that neither the sentence itself nor its negation is epistemically true, i.e., neither the sentence nor its negation is believed, known, asserted, assented to or taken as true in the imagination. So not only does the law of contradiction not hold in epistemic worlds, but also the law of the excluded middle is not sound in epistemic worlds and should not be a law of imaginary logic, if this logic is treated as a kind of epistemic logic.

With the interpretation of imaginary worlds as epistemic worlds we are in agreement with Vasiliev's intuition concerning the worlds of imaginary logic. Vasiliev determines imaginary worlds as worlds of our imagination, that can exist in our consciousness, even if they cannot exist in reality. The worlds of imaginary logic are epistemic worlds, imagined worlds but not real worlds. The realm of imaginary logic does not comprise the ontologically possible worlds, but epistemically possible worlds.

In epistemic worlds, epistemically positively characterized sentences are true, that are believed, accepted, assented to, asserted etc. If we search for an adequate logical entailment relationship for epistemic truths, then we should

look for an entailment relationship that leads from epistemic true sentences once again to epistemic true sentences, at least in a dispositional epistemic sense. Such a disposition could be that the epistemic subject assents to an epistemically logically entailed sentence in every case in which this subject has to decide whether to assent to this sentence or not. The dispositions of this kind that are fulfilled by an epistemic subject depends on the internal logical abilities of this epistemic subject and are not determined by external logical entailment relations. If epistemic truth is closed under such an epistemic entailment relation it is a kind of dispositional epistemic truth. This dispositional epistemic truth has to be discerned from actual epistemic truth (sentences which are actually believed or asserted, assented to etc.) and (in the sense of a presupposed logical entailment relation S) implicit epistemic truth (sentences which follow from epistemic truths according to S). Both, actual and dispositional epistemic truths are kinds of explicit epistemic truths in contrast to implicit epistemic truths.

Unlike classical logic, imaginary logic—as the logic of explicit epistemic truth—cannot be based on the assumption of the impossibility that a sentence and its negation are epistemically true and thus that neither of them is epistemically true. According to this, in imaginary logic not only the law of contradiction but also the law of excluded middle is not a sound law. Additionally, in an adequate imaginary logic all those logical principles should be excluded, presupposing that a sentence and its negation cannot both be true and that one of them has to be true. This situates imaginary logic in the neighborhood of the system of tautological entailments [Anderson & Belnap 1975, §§ 15, 19], [Anderson, Belnap *et al.* 1992, §§ 80 f.].

All formal differences between Aristotelian and imaginary logic are connected to formal differences in the treatment of negation in each of these logics. As pointed out by Vasiliev, we have different negations in Aristotelian and imaginary logic:

Because the law of the contradiction is a result from the definition of the negation, to build up a logic free of the law of the contradiction indicates to build up just such a logic in which our negation, which is led back on the incompatibility, does not exist. Here the imaginary logic also begins. Its method consists in the construction of another negation than ours, in the generalization of the concept of the negated judgment. [Vasiliev 1912, 62]

Vasiliev applies his considerations about imaginary logic to traditional syllogistics and demonstrates the soundness of special syllogistic principles in imaginary logic. However, he does not apply his ideas about imaginary logic to modern Fregean logical systems. In particular he gives no hints concerning the results of the application of his views to contradiction and negation in the field of propositional logic.

The following considerations were aimed as an attempt to implement Vasiliev's ideas into the construction of fitting semantic systems for propositional logic, to explain logical entailment relations for explicit epistemic truth.

2 Basic considerations for epistemic semantics

The basic differentiation that is important for epistemic semantics is the difference between epistemic and ontological worlds. Together these kinds of world are taken to be all the worlds considered for the establishment of ontological and epistemic logics. These logics are founded on the truth of sentences or principles in possible ontological or epistemic worlds:

A principle P is (ontological, epistemic) logically sound
if and only if
 P is true in all possible (ontological, epistemic) worlds.

Between possible worlds we have the following relations:

If W is the set of possible worlds, O is the set of ontological possible worlds and E is the set of epistemic possible worlds, then we have:

$$W = O \cup E \text{ and } O \cap E = \emptyset.$$

Worlds are considered as sets of sentences. A sentence G is true in a possible world w exactly when G is an element of the set w :

$$v(G, w) = t \iff G \in w.$$

The sentence G is false in a possible world w exactly when G is not an element of the set w :

$$v(G, w) = f \iff G \notin w.$$

While worlds w from O are complete (for every G , G is true in w or $\sim G$ is true in w) and consistent (it is not the case that both G and $\sim G$ are true in w), not all worlds from E are complete and not all worlds from E are consistent: Thus we have an accordance with the treatment of Vasiliev's imaginary worlds as epistemic worlds. In such epistemic worlds neither the law of contradiction nor the law of the excluded middle holds.

We next started to build up variants of epistemic semantics for the logic of assent. These epistemic semantics for the imaginary worlds and the corresponding logic of (dispositional) assent will be maximally orientated on the classical semantics: These epistemic semantics are maximal in the sense that they are transformed into a semantics for ontological worlds (adequate for classical logic) if the following bridge principles, which are not sound in the epistemic semantics, are added to the epistemic semantics:

$$\begin{aligned} (\text{cons}) \quad & v(G, w) = t \Rightarrow \neg v(\sim G, w) = t \\ (\text{comp}) \quad & \neg v(G, w) = t \Rightarrow v(\sim G, w) = t. \end{aligned}$$

In constructing epistemic semantics for logics of assent we adopted the following basic principles:

- P1 Because for all ontological worlds the principles of classical semantics are sound and all ontological worlds are epistemic worlds, the epistemic semantics should only contain sound principles taken from classical semantics. No classically unsound semantic principle is an epistemically sound principle. The epistemic semantics are partial systems of classical ontological semantics.
- P2 Because of the possibility of incomplete epistemic worlds we acknowledge that from the epistemic falsity of a sentence (the absence of a sentence in an imaginary world) it cannot be concluded that another sentence is contained in this epistemic world. Particularly, it cannot be concluded that the negation of an epistemically false sentence is epistemically true. Principle P2 is a generalization of the unsoundness of the law of the excluded middle in epistemic worlds. With P2 the principle of epistemic completeness (comp) is rejected. However the rejection of (comp) is merely a partial case of the application of principle P2. In a complete formulation P2 states: from the fact that for an epistemic world w expressions of the kind $\neg v(G, w) = t$ are among the premises concerning this world, the conclusion that another sentence H is contained in this world w cannot be drawn, if the premises are classically consistent and $v(H, w) = t$ cannot be drawn as a conclusion from those premises that don't have the form $\neg v(G, w) = t$.

A semantic principle of the kind

$$\alpha_1, \dots, \alpha_n \Rightarrow v(H, w) = t,$$

where the $\alpha_1, \dots, \alpha_n$ are composed from expressions of the form $v(G, w) = t$,

can be sound for epistemic worlds

only if this principle is sound for ontological worlds and there is no set of semantic expressions $\{\beta_1, \dots, \beta_m\}$ such that for all α_i ($1 \leq i \leq n$) holds $\beta_1, \dots, \beta_m \Rightarrow \alpha_i$ and it does not classically hold

$$\beta_{i1}, \dots, \beta_{ik} \Rightarrow v(H, w) = t,$$

where $\{\beta_{i1}, \dots, \beta_{ik}\}$ is the set of those elements from $\{\beta_1, \dots, \beta_m\}$, which don't have the form $\neg v(G, w) = t$.

Postulate P2 not only rules out classical semantic principles concerning the use of negation as in the special case (comp), but also has considerable consequences for other classical semantic principles.

So the classical principle:

$$(v(G, w) = t \Rightarrow v(H, w) = t) \Rightarrow v(G \supset H, w) = t$$

is ruled out, because we have

$$\neg v(G, w) = t \Rightarrow (v(G, w) = t \Rightarrow v(H, w) = t).$$

From this principle we acquire

$$\neg v(G, w) = t \Rightarrow v(G \supset H, w) = t,$$

and this is ruled out by P2, because omitting premises $\neg v(G, w) = t$ we don't have a classically sound semantic principle. According to the ruled out principle, the containment of any implication with the antecedent H in an epistemic world would follow from the non-containment of this antecedent H in the epistemic world. P2 excludes such irrelevant conclusions.

According to P2 the well-known classical principle for disjunction splitting

$$v(G \vee H, w) = t \Rightarrow v(G, w) = t \vee v(H, w) = t,$$

is not acceptable for imaginary worlds because this is equivalent to

$$v(G \vee H, w) = t \ \& \ \neg v(G, w) = t \Rightarrow v(H, w) = t,$$

and omitting $\neg v(G, w) = t$ we obtain the classically unsound principle

$$v(G \vee H, w) = t \Rightarrow v(H, w) = t.$$

P3 Imaginary worlds can contain arbitrary sentences. From the containment of a sentence or a set of sentences in an imaginary world, the exclusion of other sentences from this world does not follow. For epistemic worlds only such principles that don't follow from principles of the following kind are sound:

$$v(G_1, w) = t \ \& \ \dots \ \& \ v(G_n, w) = t \Rightarrow \neg v(H, w) = t.$$

This postulate is a generalization of Vasiliev's rejection of the law of contradiction for imaginary worlds. As a specification from P3 the unsoundness of the epistemic consistency principle follows:

$$(\text{cons}) \quad v(G, w) = t \Rightarrow \neg v(\sim G, w) = t.$$

But also, other classically sound principles are excluded, such as

$$v(G \vee H) \Rightarrow \neg v(\sim G, w) \vee \neg v(\sim H, w) = t,$$

because the equivalent principle

$$v(G \vee H) \ \& \ v(\sim G, w) \Rightarrow \neg v(\sim H, w) = t$$

is directly excluded by P3.

- P4 If S is a set of premises about the containment of sentences in an epistemic world w , then from this set it cannot be concluded that the epistemic world w contains arbitrary sentences. There is not epistemic world that contains every sentence.

According to this, there is no sound principle of the type

$$v(G_1, w) = t \ \& \ \dots \ \& \ v(G_n, w) = t \Rightarrow \forall p(v(p, w) = t)$$

and no sound principle of the type

$$v(G_1, w) = t \ \& \ \dots \ \& \ v(G_n, w) = t \Rightarrow v(p, w) = t,$$

where p does not occur in G_1, \dots, G_n .

From P4 for imaginary worlds the unsoundness of the contradiction explosion principle follows:

$$(CE) \quad v(G, w) = t \ \& \ v(\sim G, w) = t \Rightarrow v(p, w) = t,$$

while the unsoundness of this principle does not follow from P3 and the unsoundness of (cons).

From P4 it follows that for epistemic worlds at least one of the classical semantic principles *disjunction introduction* and *disjunctive syllogism* is not a sound semantic principle:

$$(DI) \quad v(G, w) = t \vee v(H, w) = t \Rightarrow v(G \vee H, w) = t$$

$$(DS) \quad v(G \vee H, w) = t \ \& \ v(\sim G, w) = t \Rightarrow v(H, w) = t.$$

With the application of both principles (DI) and (DS) we would receive the unsound contradiction explosion (CE) principle, excluded by P4.

In classical logic both principles are sound and so the contradiction explosion is sound in classical logic. However, there are non-classical systems, which adopt only one of the (DI) and (DS) principles: in Parry's system of analytical implication and in Zinoviev's system of strong logical entailment the principle (DI) is given up and (DS) holds, while in Anderson/Belnap's system of tautological entailment principle (DI) holds and (DS) is abandoned. However, there is a weakened variant of (DS), which holds in Tautological Entailments, namely:

$$(DS^*) \quad v(G \vee H, w) = t \ \& \ v(\sim G, w) = t \ \& \ \neg v(G, w) = t \Rightarrow v(H, w) = t.$$

The unsoundness of disjunctive syllogism in its general form in Tautological Entailments is caused by the possibility of inconsistent sentences containment in an epistemic world and the soundness of (DI). However, the disjunctive syllogism (DS) could be sound without confinement, even if we do not plainly reject (DI), as was done by Parry and Zinoviev, but confine the soundness of (DI) to applications where the antecedent is given consistently:

$$(DI^*) \quad (v(G, w) = t \ \& \ \neg v(\sim G, w) = t) \Rightarrow v(G \vee H, w) = t.$$

So, we can have (DS) in its general form, without Parry's and Zinoviev's syntactic conceptualistic attitude that no descriptive material absent from the premises can be contained in the conclusion. However, this

we got by semantically confining the disjunction introduction in the same spirit as (DS) was confined to cases, in which the premise G of disjunction introduction is given consistently, i.e., its negation $\sim G$ is not contained in the epistemic world that contains G . So, to avoid the contradiction explosion, the alternative between the system of Parry and Zinoviev on the one hand and the system of Tautological Entailments on the other hand is not exhaustive: there are other alternatives to avoid (CE), which are in the spirit of tautological entailments, but have not been treated in non-classical logic so far.

In addition to the negative postulates P2, P3 and P4 for the construction of semantics for imaginary worlds, the following positive postulate is adopted, which demands a kind of maximality for epistemic semantics:

- P5 Semantic systems appropriate for imaginary worlds are transformed into semantic systems adequate for classical logic, if the principles (cons) and (comp) are additionally assumed for epistemic worlds.

3 The epistemic semantics S1 and S2

Under the guidance of the principles P1 to P5 we formulated the following Semantics S1 and S2.¹ As primitive logical connectives in the object-language we use negation \sim and disjunction \vee and we confine the adopted basic interpretation rules to not more than three occurrences of negation \sim and to one occurrence of disjunction \vee . Conjunction \wedge , implication \supset and equivalence were introduced by definition. Then two alternative systems of epistemic semantics arose:

- $$\begin{aligned} D1. \quad (G \wedge H) &=_{df} (\sim(\sim G \vee \sim H)) \\ D2. \quad (G \supset H) &=_{df} (\sim G \vee H) \\ D3. \quad (G \equiv H) &=_{df} ((G \supset H) \wedge (H \supset G)). \end{aligned}$$

Semantics S1

- $$\begin{aligned} IR0. \quad & w \in O \Rightarrow (v(G, w) = t \Leftrightarrow \neg v(\sim G, w) = t) \\ IR1. \quad & v(\sim \sim G, w) = t \Leftrightarrow v(G, w) = t \\ IR2. \quad & v(G, w) = t \Rightarrow v(G \vee H, w) = t \\ IR3*. \quad & v(G \vee H, w) = t \ \& \ v(\sim G, w) = t \ \& \ \neg v(G, w) = t \Rightarrow v(H, w) = t \\ IR4. \quad & v(\sim(G \vee H), w) = t \Leftrightarrow v(\sim G, w) = t \ \& \ v(\sim H, w) = t \\ IR5. \quad & \forall w (v(G, w) = t \Leftrightarrow v(H, w) = t) \Rightarrow (v(F, w) = t \Leftrightarrow v(F[G/H]_{rep}, w) = t). \end{aligned}$$

1. We gave a detailed foundation for these semantics according to principles P1 to P5 in [Stelzner 2013].

Semantics S2

- $IR0.$ $w \in O \Rightarrow (v(G, w) = t \Leftrightarrow \neg v(\sim G, w) = t)$
 $IR1.$ $v(\sim \sim G, w) = t \Leftrightarrow v(G, w) = t$
 $IR2^*.$ $v(G, w) = t \ \& \ \neg v(\sim G, w) = t \Rightarrow v(G \vee H, w) = t$
 $IR3.$ $v(G \vee H, w) = t \ \& \ v(\sim G, w) = t \Rightarrow v(H, w) = t$
 $IR4.$ $v(\sim (G \vee H), w) = t \Leftrightarrow v(\sim G, w) = t \ \& \ v(\sim H, w) = t$
 $IR5.$ $\forall w (v(G, w) = t \Leftrightarrow v(H, w) = t) \Rightarrow (v(F, w) = t \Leftrightarrow v(F[G/H]_{rep}, w) = t).$

With the variant of the restriction of the introduction rule for the disjunction to internally consistent assumptions inserted in S2, we have a way to avoid the connected with the adoption of IR2 and IR3 violation of the basic principles P1 to P5. This possibility was overlooked and was not persecuted in the system of tautological entailments. As an alternative to avoid intuitively unacceptable results from the joint adoption of IR2 and IR3 in the case of unlimited acceptance of IR3 (the disjunctive syllogism), we have the flat rejection of the disjunction introduction IR2, as happens in Parry/Dunn's system of first degree analytic implication or in Sinowiew's system of strict entailment. Also, in line with the basic intuition of tautological entailments, IR2 can be restricted to IR2* to ensure that the disjunction could be introduced under the condition of the consistent adoption of one of the members of the introduced disjunction.

In both semantics S1 and S2 we have the following derived semantic principles:

- $IR6.$ $v(G \vee G, w) = t \Leftrightarrow v(G, w) = t$
 $IR7.$ $v(G \vee H), w) = t \Leftrightarrow v(H \vee G, w) = t$
 $IR8.$ $v(G \wedge H, w) = t \Leftrightarrow v(G, w) = t \ \& \ v(H, w) = t$
 $IR9.$ $v(G \supset H, w) = t \Leftrightarrow v(\sim H \supset \sim G, w) = t.$

In S1 we have:

- $IR10.$ $v(\sim G, w) = t \Rightarrow v(\sim (G \wedge H), w) = t$
 $IR11^*.$ $v(\sim (G \wedge H) \ \& \ v(G, w) = t \ \& \ \neg v(\sim G, w) = t \Rightarrow v(\sim H, w) = t$
 $IR12.$ $v(\sim G, w) = t \Rightarrow v(G \supset H, w) = t$
 $IR13^*.$ $v(G \supset H, w) = t \ \& \ v(G, w) = t \ \& \ \neg v(\sim G, w) = t \Rightarrow v(H, w) = t.$

In S2 we have:

- $IR10^*.$ $v(\sim G, w) = t \ \& \ \neg v(G, w) = t \Rightarrow v(\sim (G \wedge H), w) = t$
 $IR11.$ $v(\sim (G \wedge H) \ \& \ v(G, w) = t \Rightarrow v(\sim H, w) = t$
 $IR12^*.$ $v(\sim G, w) = t \ \& \ \neg v(G, w) = t \Rightarrow v(G \supset H, w) = t$
 $IR13.$ $v(G \supset H, w) = t \ \& \ v(G, w) = t \Rightarrow v(H, w) = t.$

4 Notions of assent in epistemic semantics

Based on the given Semantics S1 and S2 we can introduce different notions of epistemic assent, where assent to a sentence means that the epistemic subject acknowledges that this sentence is true. In contrast to the classical truth concept, the concept of epistemic truth is related to epistemic subjects. We supposed that an epistemic subject in a possible ontological world could have different possible epistemic worlds. One of these possible epistemic worlds is the actual epistemic world of the subject. The actual epistemic world contains those sentences that are acknowledged as true sentences by the epistemic subject. Needless to say, the different epistemic subjects can be connected with different possible epistemic worlds and different actual epistemic worlds. We express the relatedness between epistemic subjects and epistemic worlds by the relations R and R_r :

$R(x, w_1, w_2)$: “ x acknowledges in the world w_1 that the world w_2 is possibly an actual world”, where w_1 is a possible ontological or epistemic world and w_2 is a possible epistemic world.

$R_r(x, w_1, w_2)$: “ x acknowledges in the world w_1 that the epistemic world w_2 is part of the actual world”. The actual world in the possible world w_1 is world w_1 .

For these relations R and R_r of epistemic relatedness we suppose the following:

- R1. $R_r(x, w_1, w_2) \Rightarrow R(x, w_1, w_2)$
(All actual epistemic worlds are possible epistemic worlds.)
- R2. $\exists w_2 R(x, w_1, w_2) \supset \exists w_2 R_r(x, w_1, w_2)$
(If x acknowledges at least one world as an epistemic possible world, then x acknowledges one world as the actual world.)
- R3. $R_r(x, w_1, w_2) \ \& \ R_r(x, w_1, w_3) \Rightarrow w_2 = w_3$
(In one possible world every epistemic subject has only one actual epistemic world.)

Concerning the relations R and R_r one can ask which of the features such as seriality, reflexivity, symmetry and transitivity hold-up. To summarize, few or none of these features holds for the relations R and R_r .

1. Seriality: $\forall x \forall w_1 \exists w_2 R_{\#}(x, w_1, w_2)$. This does not hold, because there are possible worlds with subjects without the ability to acknowledge worlds as possible or actual worlds. Our real world is such a possible world.
2. Reflexivity: $\forall x \forall w R_{\#}(x, w, w)$. This does not hold, because epistemic subjects can fail to acknowledge the actual world as the actual world, or even as a possible world.

3. Symmetry: $\forall x \forall w_1 \forall w_2 (R_{\#}(x, w_1, w_2) \Rightarrow R_{\#}(x, w_2, w_1))$. This is not sound: For instance, if the possible world w_2 in the actual world w_1 is a world which is acknowledged by x as a possible world than it is possible that x in this possible world w_2 does not acknowledge the world w_1 as a possible world.
4. Transitivity: $\forall x \forall w_1 \forall w_2 \forall w_3 (R_{\#}(x, w_1, w_2) \ \& \ R_{\#}(x, w_2, w_3) \Rightarrow R_{\#}(x, w_1, w_3))$. Transitivity would imply that every epistemic subject in every actual world should acknowledge as true all those sentences, acknowledged as actual true from the standpoint of possible worlds.

With the help of the given Relations R and R_r we could determine three variants of predicates of epistemic truth for the epistemic subject x , where A is the predicator of actual assent (which holds for such sentences contained in the actual epistemic world), A_s is treated as strong assent (assented to in every possible alternative epistemic world), and A_w stands for weak assent (true in at least one world, the possibility of which is acknowledged in the given world):

$$IA1. \quad v(A(x, G), w) = t \Leftrightarrow \exists w_1 (R_r(x, w, w_1) \ \& \ v(G, w_1) = t).$$

Besides the factual notion of assent we could define the strong notion of assent $A_s(x, G)$:

$$IA2. \quad v(A_s(x, G), w) = t \Leftrightarrow \exists w_1 R(x, w, w_1) \ \& \ \forall w_1 (R(x, w, w_1) \Rightarrow v(G, w_1) = t).$$

as the truth in all imaginary worlds which are possible actual worlds for the epistemic subject x . The explicit adoption of $\exists w_1 R(x, w, w_1)$ in this interpretation rule was unavoidable, because the relation R is not serial. Without seriality it would be, that an object, which does not hold in the world w an epistemically possible world for the possibly actual world, would strongly assent in world w to any sentence. So, e.g., without condition $\exists w_1 R(x, w, w_1)$, a stone would strongly assent to any sentence, which is obviously not the case.

In addition to the strong notion of assent $A_s(x, G)$ a weak notion of assent $A_w(x, G)$ should be introduced:

$$IA3. \quad v(A_w(x, G), w) = t \Leftrightarrow \exists w_1 (R(x, w, w_1) \ \& \ v(G, w_1) = t).$$

This weak notion of assent is functionally independent of $A_s(x, G)$, but is implied by the strong notion of assent.

Adding the principles IA1, IA2 and IA3 to semantics S1 and S2 we acquire the semantics S1A and S2A.

The logical entailment relation for the given two semantics S1A and S2A can be defined in the following way:

DF. The expression H follows from the expressions G_1, \dots, G_n , in the semantics S (in symbolic form: $G_1, \dots, G_n \models_S H$, where S stands for S1A or S2A) if and only if for every interpretation in S it holds that:

$$w \in K \ \& \ v(G_1, w) = t \ \& \ \dots \ \& \ v(G_n, w) = t \Rightarrow v(H, w) = t.$$

Because of the unsoundness of $(v(G, w) = t \Rightarrow v(H, w) = t) \Rightarrow v(G \supset H, w) = t$, for S1A and S2A the deduction theorem does not hold in the following form

$$\begin{aligned} (A(x, G_1), \dots, A(x, G_{n-1}), A(x, G_n) \models A(x, H)) &\Rightarrow \\ \Rightarrow (A(x, G_1), \dots, A(x, G_{n-1}) \models A(x, G_n \supset H)). \end{aligned}$$

From the unsoundness of the deduction theorem, there is no sentence G with $\models_S A_{\#}(x, G)$. Especially, it does not hold that logically sound sentences are assented to by any epistemic subjects, so the Gödel-Rule “If $\models_S G$, then $\models_S A_{\#}(x, G)$ ” does not hold. (Here, and below $A_{\#}$ stands for A , A_s or A_w).

Finally, let us consider some sound entailment relations for S1A (abbreviated to 1) and S2A (abbreviated to 2):

- T1. $\models_S A_s(x, p) \supset A_w(x, p) \wedge A(x, p)$
- T2. $\models_S A(x, p) \supset A_w(x, p)$
- T3. $\models_S A_{\#}(x, p \wedge q) \equiv A_{\#}(x, p) \wedge A_{\#}(x, q)$
- T4. $\models_S A_{\#}(x, p \vee q) \equiv A_{\#}(x, q \vee p)$
- T5. $\models_S A_{\#}(x, p \vee p) \equiv A_{\#}(x, p)$
- T6. $\models_1 A_{\#}(x, p) \vee A_{\#}(x, q) \supset A_{\#}(x, p \vee q)$
- T7. $\models_2 A_w(x, p) \wedge \sim A_w(x, \sim p) \vee A_w(x, q) \wedge \sim A_w(x, \sim q) \supset A_w(x, p \vee q)$
 $\models_2 A(x, p) \wedge \sim A(x, \sim p) \vee A(x, q) \wedge \sim A(x, \sim q) \supset A(x, p \vee q)$
- T8. $\models_2 A_s(x, p) \wedge \sim A_w(x, \sim p) \vee A_s(x, q) \wedge \sim A_w(x, \sim q) \supset A_s(x, p \vee q)$
- T9. $\models_1 A(x, p \vee q) \wedge A(x, \sim p) \wedge \sim A(x, p) \supset A(x, q)$
- T10. $\models_1 A_s(x, p \vee q) \wedge A_s(x, \sim p) \wedge \sim A_w(x, p) \supset A_s(x, q)$
- T11. $\models_2 A_s(x, p \vee q) \wedge A_s(x, \sim p) \supset A_s(x, q)$
 $\models_2 A(x, p \vee q) \wedge A(x, \sim p) \supset A(x, q)$
- T12. $\models_2 A_s(x, p \vee q) \wedge A_w(x, \sim p) \supset A_w(x, q)$.

And for the implication:

- T13. $\models_S A_{\#}(x, p \supset q) \equiv A_{\#}(x, \sim q \supset \sim p)$
- T14. $\models_S A_{\#}(x, p \supset \sim p) \supset A_{\#}(x, \sim p)$
- T15. $\models_S A_{\#}(x, (p \supset q) \supset p) \supset A_{\#}(x, p)$
- T16. $\models_1 A_{\#}(x, \sim p) \supset A_{\#}(x, p \supset \sim p)$
- T17. $\models_2 A_s(x, \sim p) \supset A_s(x, p \supset \sim p) \vee A_w(x, p \wedge \sim p)$
 $\models_2 A(x, \sim p) \supset A(x, p \supset \sim p) \vee A(x, p \wedge \sim p)$
- T18. $\models_1 A_{\#}(x, \sim p) \vee A_{\#}(x, q) \supset A_{\#}(x, p \supset q)$
- T19. $\models_2 A_w(x, \sim p) \wedge \sim A_w(x, p) \vee A_w(x, q) \wedge \sim A_w(x, \sim q) \supset A_w(x, p \supset q)$
 $\models_2 A(x, \sim p) \wedge \sim A(x, p) \vee A(x, q) \wedge \sim A(x, \sim q) \supset A(x, p \supset q)$
- T20. $\models_2 A_s(x, \sim p) \wedge \sim A_w(x, p) \vee A_s(x, q) \wedge \sim A_w(x, \sim q) \supset A_s(x, p \supset q)$

- $T21. \models_1 A(x, p \supset q) \wedge A(x, p) \wedge \sim A(x, \sim p) \supset A(x, q)$
 $T22. \models_1 A_s(x, p \supset q) \wedge A_s(x, p) \wedge \sim A_w(x, \sim p) \supset A_s(x, q)$
 $T23. \models_2 A_s(x, p \supset q) \wedge A_s(x, p) \supset A_s(x, q)$
 $\quad \models_2 A(x, p \supset q) \wedge A(x, p) \supset A(x, q)$
 $T24. \models_2 A_s(x, p \supset q) \wedge A_w(x, p) \supset A_w(x, q).$

If the different notions of assent A , A_s and A_w are interpreted as notions of explicit assent or as notions of the disposition for explicit assent, then this interpretation is connected to the empirical assumption in these semantics epistemic subjects have perfect logical abilities concerning these semantics. Without this empirical assumption, these notions can be considered as notions of implicit assent, where two different kinds of implicitness are determined by the semantics S1A and S2A. Consequently, this implicitness is not determined by the classical entailment relation (as is usually done in standard epistemic logic).

So far, for the connection between the relations R and R_r we supposed the following:

- $(I) \quad R_r(x, w_1, w_2) \Rightarrow R(x, w_1, w_2).$
 $(II) \quad \exists w_2 R(x, w_1, w_2) \Rightarrow \exists w_2 R_r(x, w_1, w_2).$
 $(III) \quad (R_r(x, w_1, w_2) \ \& \ R_r(x, w_1, w_3) \Rightarrow w_2 = w_3).$

Now we suppose that an epistemic subject x holds that a world w is the actual one only in cases where x holds that w is a possible actual world. But it is possible that x holds that worlds are possible actual worlds, which entirely differ from the world which x holds for the actual world. What x holds for a possible world is entirely independent from what x holds for the actual world. This indicates that so far we have considered the relation between epistemically possible worlds and the epistemically actual world just in agreement with the relation between ontologically possible worlds and the ontologically actual one.

But one can suppose a closer epistemic connection between the sentences in world w_1 that are assented to by x (which represent the world recognized by x as the actual world in world w_1) and those worlds which are considered by x as possible actual worlds. We can suppose that in these worlds, considered by x as possible actual worlds, every sentence is true, which is recognized by x as true in the actual world. Following this, the world that is acknowledged by the epistemic subject as the actual world is a partial world of every epistemically possible world of this epistemic subject. According to this, with the assumption mentioned, we obtain:

- $(IV) \quad \exists w_2 (R_r(x, w_1, w_2) \ \& \ v(H, w_2) = t) \Rightarrow \forall w_3 (R(x, w_1, w_3) \Rightarrow v(H, w_3) = t).$

However it is not excluded that untrue sentences in the world considered by x as the actual world are true in worlds considered by x as possible actual worlds. From (IV) we obtain:

- $(1) \quad \exists w_1 R(x, w_1, w_2) \ \& \ R_r(x, w_1, w_2) \ \& \ v(H, w_2) = t \Rightarrow$
 $\quad \Rightarrow (\exists w_1 R(x, w_1, w_2) \ \& \ \forall w_3 (R(x, w_1, w_3) \Rightarrow v(H, w_3) = t)).$

With IA2 we obtain:

$$(2) \quad \exists w_1 R(x, w, w_1) \ \& \ R_r(x, w_1, w_2) \ \& \ v(H, w_2) = t \Rightarrow v(A_s(x, G), w_1) = t.$$

Using IA1 and (I) we obtain:

$$(3) \quad v(A(x, G), w_1) = t \Rightarrow \exists w_1 R(x, w, w_1) \ \& \ R_r(x, w_1, w_2) \ \& \ v(H, w_2) = t.$$

From (2) and (3)

$$(4) \quad v(A(x, H), w) = t \Rightarrow v(A_s(x, H), w) = t$$

So we have:

$$T25. \models_S A_s(x, p) \supset A(x, p)$$

With T1 this gives:

$$T26. \models_S A_s(x, p) \equiv A(x, p).$$

So, with the help of condition (IV) we can eliminate the strong notion of assent by the actual notion. But we cannot eliminate the weak notion of assent, which is not a kind of actual assent, but merely an indication that by excluding several epistemic alternatives we could arrive at the notion of actual assent.

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