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Minimal Logicism

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Résumé : PLV (*Plural Basic Law V*) est un système de second ordre cohérent qui vise à dériver l'arithmétique de Peano du second ordre. Il emploie la notion de quantification plurielle et une formulation du premier ordre de la tristement célèbre Loi Fondamentale V de Frege. La sémantique plurielle de George Boolos est remplacée par la *Acts of Choice Semantics* (ACS) de Enrico Martino, qui est développée à partir de la notion de référence arbitraire en raisonnement mathématique. ACS fournit une forme de logicisme qui est radicalement alternative à celle de Frege et qui est fondée sur l'existence des individus plutôt que sur l'existence des concepts.

Abstract: PLV (*Plural Basic Law V*) is a consistent second-order system which is aimed to derive second-order Peano arithmetic. It employs the notion of plural quantification and a first-order formulation of Frege's infamous Basic Law V. George Boolos' plural semantics is replaced with Enrico Martino's *Acts of Choice Semantics* (ACS), which is developed from the notion of arbitrary reference in mathematical reasoning. ACS provides a form of logicism which is radically alternative to Frege's and which is grounded on the existence of individuals rather than on the existence of concepts.

It is well-known that Frege's logicist foundation of mathematics exposed in *Grundgesetze der Arithmetik* is inconsistent. The contradiction is derived from the infamous Basic Law V. This principle is crucial to Frege's logicism as it embeds the tenet that tightly connects natural numbers, conceived as equivalence classes, to concepts.

Since, according to Frege, extensions are logically dependent on concepts, numbers as extensions inherit their logicity from that relation of logical dependence. The failure of his programme doomed the possibility of deriving arithmetic on purely logical basis, where the overall logicity of the programme was embedded in the logical connection between concepts and extensions.

It is also well-known, though, that there are indeed consistent fragments of Frege's *Grundgesetze*. In the 80s, Terence Parsons proved that the first-order fragment of Basic Law V has a model. In 1996, Richard Heck proved the semantic consistency of the predicative fragment of *Grundgesetze*; and in a few papers dated from 1999 to 2002, Kai Wehmeier and Fernando Ferreira proved the consistency of its Δ_1^1 -fragment.

These consistency proofs establish a merely technical result.¹ But they may prompt a rather natural question, namely whether these consistent fragments of *Grundgesetze* may provide a formal bulk to revise Frege's programme. This question is really two-fold: on the one hand, it concerns the mathematical strength of these fragments; on the other, it regards the possible revision of Frege's philosophical assumptions, in particular, the Fregean assumption on the existence of concepts. As for the first issue, the consistent subsystems interpret at most Robinson arithmetic. Though not a trivial result, this is slightly disappointing, especially as compared to Frege's original programme of founding full second-order Peano arithmetic on merely logical basis. The second issue is rather significant, since, as mentioned, Frege's logicism is underpinned by the relation of logical dependence of extensions from concepts.

In this article, I will argue for a *minimal* form of logicism, as captured by an axiomatisation that deploys the philosophically minimal assumptions necessary for recovering second-order arithmetic, in such a way that these assumptions may incorporate a form of logicism alternative to Frege's. I will present a formal theory that extends the consistent first-order fragment of Frege's *Grundgesetze* by adding plural logic to it. The resulting system I will call *Plural Basic Law V* (PLV). The main features of PLV are plural quantification, which guarantees the strength of full second-order logic to PLV, and a particular semantics, the *Acts of Choice Semantics* (ACS), employed to interpret first-order and plural quantification. The minimal assumptions I am here arguing for concern the lack of second-order ontological commitment and the lack of first-order metaphysical commitment, where the former concerns the existence of second-order entities, and the latter concerns what kind of entities the first-order individuals are. On the grounds of ACS, PLV embodies a minimal form of logicism, which deploys somehow very little ontological effort for recovering PA^2 , and is radically different from Frege's, as it is grounded on the existence of individuals rather than on the existence of concepts.

1 Plural BLV: a system

The basic features of the language $\mathcal{L}$ of PLV are:

- (i) an infinite list of singular individual variables $x, y, z, \dots$;

1. Possibly, that is what the authors themselves had in mind anyway.

- (ii) an infinite list of plural individual variables $xx, yy, zz, \dots$, that vary *plurally* over the individuals of the first-order domain;
- (iii) the logical constants $\neg, \rightarrow, =$;
- (iv) existential quantifiers $\exists$ for every kind of variables;
- (v) the constant relation symbol $\prec$;
- (vi) the abstraction operator $\{:\}$.

The atomic formulæ of $\mathcal{L}$ are:

- (vii) $\mathbf{a} = \mathbf{b}$;
- (viii) $\mathbf{a} \prec \mathbf{bb}$,²

where $\mathbf{a}$ and $\mathbf{b}$ are metavariables for the terms of $\mathcal{L}$, and $\mathbf{bb}$ is a metavariable for plural variables. Formulæ of kind (viii) express what I may call *plural reference*. Primitive existential quantification for both kinds of variables is available. Universal quantification can be defined in the obvious way. Along with the singular variables $x, y, z, \dots$, the *terms* of $\mathcal{L}$ are:

- (ix) an infinite list of extension-terms of the form $\{x : \psi x\}$,

where ψ is a first-order formula of $\mathcal{L}$, i.e., a formula not containing plural variables at all. It may contain, though, both free and bound singular variables. Also, nested extension-terms may appear in extension-terms. A comprehension principle that governs pluralities is available in PLV:

$$(PLC). \exists yy \forall x (x \prec yy \leftrightarrow \phi x),$$

where ϕ does not contain yy free; and a schematic formulation of first-order *Basic Law V*:

$$(V). \{x : \psi x\} = \{x : \chi x\} \leftrightarrow \forall x (\psi x \leftrightarrow \chi x),$$

where ψ and χ do not contain plural variables at all. Axiom V guarantees the existence of Dedekind-infinitely many first-order individuals in the domain. This is crucial to guarantee that Peano axioms may be derived in PLV. Later on in the article, I'll say more about the restrictions on plural variables in extension-terms.

2 Peano axioms

A few more definitions are needed in order to derive Peano axioms. The *singleton* and the notion of *unordered pair* may be defined as usual:

2. To be read “ $\mathbf{a}$ is among the $\mathbf{bs}$ ”.

Definition 1. $\{x\} =_{def} \{y : x = y\}$;

Definition 2. $\{x, y\} =_{def} \{z : z = x \vee z = y\}$.

The usual Wiener-Kuratowski definition of the *ordered pair* easily follows:

Definition 3. $(x, y) =_{def} \{\{x\}, \{x, y\}\}$.³

Notice that, strictly speaking, $\mathcal{L}$ is monadic. The introduction of pairs, nevertheless, provides $\mathcal{L}$ with *polyadic* expressive capacity: the formula $(x, y) \prec zz$, in fact, means that the individual (x, y) is among the individuals zz . In $\mathcal{L}$, *natural numbers* may be defined inductively. The individual constant “0” may be introduced by definition:

Definition 4. $0 =_{def} \{x : x \neq x\}$.

Consequently, numbers may be inductively defined:

Definition 5. $1 =_{def} \{x : x = 0\}$;

Definition 6. $2 =_{def} \{x : x = 1\}$;

and so on. In general, the *successor* of a number is its singleton. In this way, we get the usual Zermelo natural numbers. A plurality xx is *inductive* whenever it contains 0 and it is closed under the successor. The usual definition of the set of natural numbers may be given in terms of pluralities. First, a plurality nn is defined:

Definition 7. $x \prec nn \leftrightarrow_{def} \forall yy (yy \text{ is inductive} \rightarrow x \prec yy)$.

Given the previous definitions, the following formulations of second-order Peano axioms are derivable in PG, with the singular variables x and y restricted to nn :

Theorem 1. $0 \prec nn$

Proof. That 0 is a number trivially follows from the definition of nn . □

Theorem 2. $\forall x (\{x\} \neq 0)$

Proof. Let us assume that there is an individual y such that $\{y\} = 0$. On the grounds of the definition of 0, thus, y must satisfy the condition $x \neq x$. As no individual is not self-identical, 0 is no successor.⁴ □

3. The fundamental law of the ordered pair $(x, y) = (u, v) \leftrightarrow x = u \wedge y = v$ may be easily derived in PLV, through several applications of the usual rules of inference, axiom V, and the definitions of the unordered and ordered pair.

4. The formal proof of this theorem makes a crucial use also of axiom V and of the definition of the singleton, among other logical resources. See [Boccuni 2013, 1054] for such a proof.

Theorem 3. $\forall xy(\{x\} = \{y\} \rightarrow x = y)$

Proof. Let x and y be two arbitrary individuals of the first-order domain of $\mathcal{L}$. If $\{x\} = \{y\}$, then, on the grounds both of axiom V and of the definition of the singleton, for all z , $z = x \leftrightarrow z = y$. Thus, for the transitivity of identity, $x = y$. As x, y are arbitrary, the generalization $\forall xy(\{x\} = \{y\} \rightarrow x = y)$ is valid. $\square$

Theorem 4. $\forall xx(0 \prec xx \wedge \forall x(x \prec xx \rightarrow \{x\} \prec xx) \rightarrow \forall x(x \prec xx))$

Proof. It trivially follows from the definition of nn . $\square$

The consistency of PLV follows from [Boccuni 2011]. The fact that PLV is consistent, though, should not be very surprising since any interaction of the problematic axioms, PLC and V, is avoided. Furthermore, that PLV interprets PA^2 follows straightforwardly from PLC and V: by PLC, we can define an inductive plurality by which we may easily recover a derivation of the axiom of full mathematical induction; by V, PLV implies the existence of enough first-order individuals for natural numbers to be defined. The interest of PLV, though, lies somewhere else, in particular in that PLV embodies the philosophically minimal assumptions we have to make in order to interpret PA^2 . These assumptions require at most the existence of infinitely many first-order individuals, which nevertheless are not to be intended as extensional entities. More precisely, they will be metaphysically inert: their essence will be completely irrelevant. The ontological and metaphysical innocence of PLV will be achieved by ACS. ACS will be introduced in the following sections, but, in order to make sense of it, a slight detour through the notion of arbitrary reference is required.

3 The theory of ideal reference

According to Martino, the possibility of directly referring, at least ideally, to any object of a universe of discourse is presupposed both by logical and mathematical reasoning, even when non-denumerable domains are concerned [Martino 2001, 2004]. Such a possibility of direct reference is very well expressed by the crucial role *arbitrary reference* plays both in formal and informal reasoning. Its cruciality lies in that arbitrary reference exhibits two different logical features that make it essential for performing proofs, i.e., *arbitrariness* and *determinacy*:

(...) The distinction between asserting $\phi(x)$ and asserting $(x)\phi x$ was, I believe, first emphasized by Frege (1893, p. 31). His reason for introducing the distinction explicitly was the same which had caused it to be present in the practice of mathematicians, namely,

that deductions can only be effected with real variables, not with apparent variables. In the case of Euclid's proofs, this is evident: we need (say) some one triangle ABC to reason about, though it does not matter *what* triangle it is. The triangle ABC is a real variable; and although it is any triangle, it remains the *same* triangle throughout the argument. [Russell 1967, 156–157, emphasis added]

Through arbitrary reference, then, we may consider *any* object a of a universe of discourse. Consequently, the arguments about a retain their general validity. At the same time, though, within the arguments about it, “ a ” is required to denote a determinate object, which stays the same throughout the derivation and is distinct from all the other objects in the domain it belongs to.

In order to motivate this claim, first of all the genuine referentiality of arbitrary names has to be accounted for. In [Boccuni 2013], a general argument for viewing arbitrary reference as genuinely referential is provided. According to this argument, the soundness of arguments in mathematical and logical reasoning is based on the underlying assumption of the genuine referentiality of arbitrary reference, where the relation between soundness and referentiality is spelled out in terms of sameness and determinacy of reference. But even if genuine referentiality of arbitrary reference were granted, though, it may still sound at odds with arbitrariness: What does it mean that we can refer to an arbitrary individual? I take that this question comes down to the issue of what is arbitrary in arbitrary reference. I see three possible ways of dealing with this issue. One possible way is to claim that what is arbitrary is the reference relation itself. For instance, according to Russell,

If we say: “Let ABC be a triangle, then the sides AB and AC are together greater than the side BC ”, we are saying something about *one* triangle, not about *all* triangles; but the one triangle concerned is absolutely ambiguous, and our statement consequently is also absolutely ambiguous.⁵ [Russell 1967, 156–157]

When Russell speaks of ambiguous names, he seems to have in mind that reference is ambiguous. Nevertheless, “ a ” has to refer to a determinate individual within an argument on a , so the reference relation between “ a ” and a , once established, is not ambiguous at all.

A rather different argument is by Kit Fine. According to him, arbitrariness is a property of some special kind of objects, namely those referred to by arbitrary names. To this extent, we may claim that, though a is an object having the property of being arbitrary, we may still determinately refer to it. Nevertheless, it is because of a property that makes a what it is, that we cannot say which object a is. Thus, a is *intrinsically* indeterminate, namely

5. “ ABC ” is a free variable.

it is indeterminate by its own nature. This would clearly violate the requirement of a being a *determinate* object, which is indeed a crucial feature of arbitrary reference.

The best way to view arbitrariness, I think, is as an *epistemic* feature of reasoning: a is determinate, and “ a ” determinately refers to it, but we do not *know which* individual a is.⁶ This interpretation, on the one hand, retains the intuition concerning generality. In a sense, our lack of knowledge of which individual a is justifies the applicability of the rule of introduction of the universal quantifier (under the usual restrictions): since a is not an individual I could pick among all others because I do not know which one it is, the conclusion I draw on a is valid for all individuals of the domain (provided that the restrictions on the rule are respected). The epistemic interpretation of arbitrariness also preserves genuineness, since I may not know which individual a is, but this is not incompatible, unlike Russell’s and Fine’s interpretations of arbitrariness, with a being a determinate individual and thus reference to a being genuine.

As I mentioned in the opening of this section, Martino claims that arbitrary reference is *direct*. In order to motivate this claim, consider the rule of existential elimination. As Martino points out, the possibility of passing from a purely existential assumption such as $\exists x\phi x$ to the consideration of an arbitrary object a such that ϕa is guaranteed by the rule of elimination of the existential quantifier which allows to substitute the given existential assumption with the auxiliary assumption ϕa [Martino 2004]. If the rules of inference that govern the use of the logical constants are justified by the meaning of the constants themselves, the meaning of the existential quantifier presupposes the possibility of singularly referring, at least ideally, to any individual, and consequently existential *quantification logically presupposes such a possibility of reference*.⁷ Thus, before we simultaneously consider several entities through quantification, we are required to be able to directly refer to each of them, at least ideally: quantification logically presupposes the ideal possibility of referring to each and every element of a domain, before we consider those elements through generalisation. [Martino 2001, 2004] label this claim the *Thesis of Ideal Reference* (TIR).

From the perspective of the logical presupposition of reference from quantification at least in an ideal way, reference to an entity exclusively in terms of reference to a totality that entity belongs to can not be allowed, because it is required that we are able to directly refer to that entity, even if just in an ideal way, on pain of violating TIR. As a corollary of TIR, in fact, Martino provides a re-formulation of Russell’s well-known *Vicious Circle Principle*

6. See [Breckenridge & Magidor 2010], [Martino 2001, 2004] on the epistemic interpretation of arbitrary reference.

7. Analogously as far as the rule of introduction for universal quantification is concerned. See [Martino 2004, 110].

(VCP*). No universe of discourse can contain an element which we can refer to only through reference to that universe.⁸

In order to make sense of such a possibility of reference, then, a theory of *direct* arbitrary reference is needed. This may remind one of Kripke's theory of direct reference in natural language. In Kripke's picture, when we attach a name to a thing, we do so by imposing a name on that thing by a ceremony of baptism, which is performed *in praesentia*. Nevertheless, since we are here dealing with mathematical entities, we cannot appeal to a casual connection to motivate direct reference as the usual Kripkean theory of reference does. To this extent, the direct theory of reference here at stake has to be a theory of *ideal* direct reference, such that it mirrors Kripke's intuitions and nevertheless is idealised in such a way to account for the semantics of mathematical discourse.

The final step that needs to be taken in this setting concerns how exactly this view of arbitrary reference should be put to work in an appropriate semantics. Such a semantics, then, would have to account for arbitrariness and directness of arbitrary reference at the same time. To this aim, Martino helps himself to a rather strong *idealisation*, which indeed makes sense of both these features. Such a strong idealisation, though quite articulate, will show to have several advantages as far as PLV is concerned. But for now let us focus on its formulation.

4 The acts of choice semantics

In order to justify the possibility of direct ideal reference, Martino proposes to imagine a series of *ideal agents* that fix the reference of the meaningful expressions of a language. The ideal agents, that are pictured as having direct access to the individuals of the universe of discourse, perform an arbitrary *act of choice* through which the reference of the meaningful expressions is fixed. This idealisation mirrors Kripke's picture of the baptism performed *in praesentia*. We can picture agents as holding scoring paddles bearing "1" on one side of the paddle, and "0" on the other. In general, as long as singular reference is concerned, for each individual a of the domain, there is an agent that picks a as the referent of " a " whenever she chooses "1" relative to " a "; the agent does not pick a as the referent of " a " whenever she chooses "0". Clearly, there have to be as many agents as individuals; but then again, since agents are mere idealisations, there is no domain of agents at all. Even more so, we may take the first-order individuals themselves to play the role of agents. The postulation of the ideal existence of the agents is just

8. [Martino 2004, 119, En. transl. mine]. Notice that VCP* follows from TIR also when non-denumerable domains are concerned. Even though a language may lack non-denumerably many names, TIR still holds, as the ideal possibility of directly referring to each and every individual in a non-denumerable domain may be performed via arbitrary reference, as in the case of, e.g., "let a be an arbitrary real number".

aimed to explain how acts of reference are performed in a formal language, see [Martino 2004, 112–113].

4.1 Plural ACS

Recall that in PLV ACS is deployed for interpreting both first-order and plural quantification. This provides a uniform semantics for PLV, but is also motivated by a further reason. Those who work in plural logic, in fact, may wonder why I am not here appealing to *Boolos' plural semantics* [Boolos 1984, 1985], in order to interpret to plural bit of PLV. Nevertheless, I think that there are reasons for unsatisfaction with it.

In [Boolos 1984, 1985], an interpretation of second-order logic that is alternative to the standard set-theoretic one is provided. Boolos' plural interpretation is grounded on the use of primitive second-order existential quantification $\exists X \dots X \dots$, which is interpreted as 'There are some X 's such that $\dots X \dots$ '. Revising Tarski's semantics, Boolos provides the semantic clauses for second-order logic with plural quantification, in which he substitutes the usual function of assignment with a *one-many relation* of assignment R . R correlates only one individual to each first-order variable, meanwhile it is not restricted as long as second-order variables are concerned. R thus may correlate zero, one or several individuals to second-order variables. The Tarski-style clauses are thus provided. The following are the relevant ones, where s is a sequence of individuals:

- (a) R and s satisfy the formula Xx if, and only if, $R < X, s(x) >$;
- (b) R and s satisfy the formula $\exists X \mathbf{A}$ if, and only if, $\exists Y \exists T (\forall x (Yx \leftrightarrow T < X, x >) \wedge \forall Z (Z \text{ is a second-order variable} \wedge Z \neq X \rightarrow \forall x (T < Z, x > \leftrightarrow R < Z, x >)) \wedge T \text{ and } s \text{ satisfy } \mathbf{A})$.

As the notion of value of a variable is supposed to be plainly captured by the definition of assignment in the metalanguage and the above conditions (a) and (b) apparently display no reference to classes, Boolos' semantics is not *per se* ontologically committed to the existence of higher-order entities. Nevertheless, Boolos' semantics has been subject to several criticisms, see [Resnik 1988], [Parsons 1990], [Linnebo 2003]. These criticisms are basically grounded on the fact that in Boolos' semantics plural reference through second-order variables $X, Y, Z, \dots$ is taken as a *primitive*, and the notion of plurality is far from being ontologically transparent. How else, in fact, would we interpret primitive plural reference, if not as reference to some kind of entities, namely pluralities, that are not, all in all, very different from classes?⁹

In order to account for Boolos' intuition, then, a semantics that does not take plural reference as a primitive is required. This is what Martino's *plural ACS* provides through the notion of act of *simultaneous* choice. By an act of

9. On this last point, see [Linnebo 2003].

simultaneous choice it is meant a simultaneous choice between the values 0,1 performed by each agent. In this way, each agent performs a merely singular choice, meanwhile the simultaneousness guarantees that such acts involve several individuals at once. An individual is, then, designated in an act of simultaneous choice, whenever the corresponding agent chooses 1 in the relative act of choice; it is not designated otherwise.

ACS then is used in order to provide the truth-clauses for the formulæ of $\mathcal{L}_{PLV}$ containing singular and plural quantifications. Let $\mathcal{D}$ be a non-empty domain of individuals. For each term t_i , consider a singular choice t_i^* of an individual of $\mathcal{D}$, for $i = 1, \dots, n, \dots$; for each plural variable xx_j , consider a simultaneous plural choice xx_j^* of individuals of $\mathcal{D}$, for $j = 1, \dots, m, \dots$. The truth-clauses for singular and plural quantifications are inductively given, then, in terms of the acts of choice $t_1^*, \dots, t_n^*, \dots, xx_1^*, \dots, xx_m^*, \dots$

Let **a**, **b**, and **c** be metavariables for the singular terms of $\mathcal{L}_{PLV}$, namely metavariables for first-order variables and extension-terms of $\mathcal{L}$; **aa** and **bb** metavariables for plural variables; and **B** a metavariable for the formulæ of $\mathcal{L}_{PLV}$. The following are the relevant inductive truth-clauses for the sentences of $\mathcal{L}_{PLV}$:

1. **a** = **b** is true iff the individual designated by the choice **a**^{*} is identical with the individual designated by the choice **b**^{*}, with respect to **c**₁^{*}, ..., **c**_n^{*} for any terms **c**₁, ..., **c**_n possibly in **a** and **b**;
2. **a** < **aa** is true iff the individual designated by the choice **a**^{*} is among the individuals designated in the plural choice **aa**^{*}, with respect to **c**₁^{*}, ..., **c**_n^{*} for any terms **c**₁, ..., **c**_n possibly in **a**;
3. $\exists \mathbf{aB}$ is true iff, corresponding to the variable **a**, it is possible to perform a singular choice **a**^{*} such that **b**₁^{*}, ..., **b**_n^{*}; **aa**₁^{*}, ..., **aa**_m^{*}; **a**^{*} $\models$ **B** for any terms **b**₁, ..., **b**_n and free plural variables **aa** possibly in **B**;
4. $\exists \mathbf{aaB}$ is true iff, corresponding to the variable **aa**, it is possible to perform a plural choice **aa**^{*} such that **a**₁^{*}, ..., **a**_n^{*}; **bb**₁^{*}, ..., **bb**_m^{*}; **aa**^{*} $\models$ **B**, for any terms **a**₁, ..., **a**_n and free plural variables **bb**₁, ..., **bb**_n possibly in **B**, see [Martino 2004, 103–133], also for the act of choice clause for the formulæ of the form $\forall \mathbf{aaB}$.

It has to be kept in mind that acts are not entities, but exactly acts. Recall that ACS is based on Kripke's intuition of how reference is fixed and works in the natural language. Thus, ideal acts can be conceived as idealisations of actual acts of reference, just as ideal agents can be conceived as idealisations of actual agents. This analogy with actual acts provides a way to make sense of how we can conceive reference to be fixed in formal languages. The quantification on acts in the previous semantic clauses, thus, is to be meant potentially. There is a substantial difference between performed acts and merely potential acts, capable of being performed by the agents. Truth-clauses for singular and plural quantifications do not “refer to a totality of acts, conceived as entities existing

in a mysterious realm: (...) as acts are not entities, it makes no sense to talk of a totality of acts" [Martino 2004, 131, En. trans. mine]. Thus, the notion of possibility in (3) and (4) implies that, among different potential acts of choice, one, either singular or plural, may be performed such that it verifies **B**. It is absolutely determinate that the agents may perform a simultaneous choice, i.e., a combination of 0,1, such that it verifies **B**. Thus, the arbitrariness with which 0 or 1 are chosen by each agent does not refute the validity of the Principle of the Excluded Middle. ACS is plainly compatible with classical logic. Although the choice between 0,1 is arbitrary, it is immediately determinate which the outcomes of any act of choice are. In fact, given some arbitrarily chosen individuals yy and an arbitrarily chosen individual x , whether x is (or is not) among the yy is an immediate outcome of which individuals yy are chosen.

5 Minimal logicism

In the present section, I shall explore the philosophical features of PLV as a minimal form of logicism. By minimal, I mean that the ontological and metaphysical assumptions underlying PLV are the least assumptions we need to recover PA².

First of all, we may accomplish ontological innocence of the plural bit of PLV through plural quantification as interpreted by ACS. Plural variables, in fact, are interpreted as varying over the first-order domain. Thus, PLC defines pluralities by quantification over pluralities, but this does not introduce a new entity, e.g., the plurality X , on the grounds of a totality it belongs to. It just indicates a multiplicity of individuals that we already have at disposal. Plural quantification is just a linguistic tool to talk about those individuals in a way which is not available to regular first-order quantification. On the other hand, given the ontological innocence of plural quantification, the impredicativity involved in PLC is consistent both with TIR and VCP*. For the very same reason, we may also allow free plural variables in PLC.

The notion of plurality, though, has been subject to the criticism that the talk of pluralities is just talk of classes in disguise—or class-like entities. This criticism, nevertheless, assumes tacitly that pluralities are entities of some sort, which instead should be firmly rejected. The talk of pluralities is just a *façon de parler*, involving no higher-order entities but only regular first-order individuals plurally considered. ACS shows this clearly, since the notion of plural reference is explained in terms of the notion of simultaneous acts of singular choice. Moreover, acts are not entities, so in ACS there is no hidden ontological commitment other than the first-order.

In the previous paragraphs, I motivated the claim that PLV is ontologically innocent as for plural quantification. In what follows, I will provide motivation for claiming that PLV's first-order fragment is *metaphysically* innocent. By the notion of metaphysical innocence, I mean that the first-order fragment

of PLV interpreted by ACS, though ontologically committed to the existence of infinitely many first-order individuals, is not committed to the existence of individuals with a peculiar nature. In particular, I claim that the individuals that extension-terms take as values need not be considered as *extensions*, i.e., as intrinsically extensional or even set-theoretical objects, but may be considered as individuals deprived of any intrinsic nature. Through ACS, the notion of satisfaction is given in terms of arbitrary choices. So, for an individual x to satisfy a formula ϕ means just to be chosen by an arbitrary choice to satisfy ϕ , without appealing to x having the property allegedly expressed by ϕ or being an element of the class allegedly individuated by ϕ . First-order individuals, then, are not conceived as the bearers of properties on the grounds of which they are distinguished from one another. The minimal condition of distinguishability of an individual from another is satisfied through the possibility of choosing and, thus, of naming that very individual instead of another. ACS, then, provides grounds for the metaphysical innocence of first-order quantification. Consider, in fact, extension-terms. On Frege's view, extensions owe their logical status to their relation of logical dependence from concepts. In PLV, the logical role that Frege assigned to concepts and their relation to extensions are not available. But then again consider that in PLV the referents of extension-terms are fixed by ACS. Through ACS, an individual is assigned to the term " $\{x : \phi x\}$ " not because that very individual is the extension of all x such that ϕ , rather because such an individual has been arbitrarily chosen as the semantic value of " $\{x : \phi x\}$ ". Thus, though PLV is indeed committed to the existence of infinitely many first-order individuals, it is *not* committed to the existence of *intrinsically* extensional objects. For this reason, the Julius Caesar problem is easily solved in PLV, since, if Julius Caesar is in PLV's domain, then it is capable of being chosen as the semantic value of a singular term in an arbitrary act of choice. So, Julius Caesar may, for instance, play the role of the empty extension, if an ideal agent chooses him to be the semantic value of "0". This solves an issue posed by Wehmeier concerning the feasibility of a logicist programme, namely that Fregean systems imply the existence of infinitely many *non*-logical objects:

one might argue that the provability of the existence of infinitely many objects other than logical ones is a *reductio ad absurdum* of a logicist system. [Wehmeier 1999, 326].

In this respect, PLV's first-order fragment is *metaphysically* innocent, since by ACS it is not committed to a sort of objects that are intrinsically set-theoretical nor intrinsically of some other kind. Axiom V, then, claims a completely arbitrary correspondence between formulæ and objects: a certain formula is not connected to an object because this latter is the extension of all objects that satisfy that formula, rather it is connected to an object which, once it has been chosen as the semantic value of a given extension-term, *plays the role* of the extension of the objects satisfying the formula.

Finally, if the argument about the ontological innocence of plural quantification through ACS holds, this also provides motivation for disallowing plural variables to interact in any way with extension-terms. In fact, extension-terms do not refer to intrinsically set-theoretic objects. So a correspondence between pluralities and single entities like first-order individuals, though consistently restricted, would sound unmotivated and possibly counterintuitive as for the intuitions we have about pluralities.

6 Conclusion

In the present article, I presented the plural system PLV, which interprets second-order Peano arithmetic (sections 1 and 2). The main features of PLV are a plural comprehension axiom and first-order Basic Law V, and ACS by Martino. ACS is motivated starting from some independent considerations about arbitrary reference in mathematical and logical reasoning. The two main issues concerning arbitrary reference are its genuine referentiality and its directness (sections 3 and 4). The very notion of arbitrary reference is then applied to first-order and plural quantification in PLV (section 4). Through ACS, arbitrary reference provides a way to motivate that PLV embodies a minimal form of logicism, namely the recovery of PA^2 from minimal ontological and metaphysical assumptions (section 5).

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