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# Is Church's Picture of Frege a Good One?

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**Résumé :** A. Church a beaucoup contribué à la sauvegarde de la théorie de la signification de G. Frege bien que des antinomies aient été découvertes en son sein. Church a dû pour cela adapter sa théorie en gardant certaines parties, en supprimant d'autres et en en ajoutant de nouvelles, la plupart exposées de manière informelle dans l'introduction de son *Introduction to Mathematical Logic*. Comme pour toute reprise d'une théorie par un autre penseur, il est intéressant de comprendre dans quelle mesure la pensée du premier subsiste malgré d'importants changements dans la nouvelle théorie, ou bien au contraire dans quelle mesure cette nouvelle pensée ne conserve que l'apparence de l'ancienne. Nous étudierons donc chaque concept fondamental de la logique que Church introduit à cette occasion et nous évaluerons ses rapports avec la théorie originale de Frege.

**Abstract:** A. Church has contributed a lot to the safeguard of G. Frege's theory of meaning after the discovery of antinomies in it. To achieve this he has adapted it by keeping parts, discarding others and adding new ones, most of which are clearly exposed in an informal way in the introduction to his *Introduction to Mathematical Logic*. As for any modification of a theory by another person, it is interesting to understand how the thoughts of the former survive in the new theory even though radical changes have occurred, or on the contrary how it retains only the appearance of the initial theory. We shall therefore go through the basic concepts of logic he introduces in this introduction and assess their relation to Frege's original theory.

## 1 Introduction

Alonzo Church (1903-1995) was an American mathematician and logician and "Frege's most powerful exponent" [Burge 1979]. He published in 1956 an influential textbook, *Introduction to Mathematical Logic*, which was intended for

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beginners to learn logic and for advanced people to use as a reference. The content is therefore structured into didactic loops: the start point (§00 of the introduction) is an answer to What is logic? Church then takes the reader through the basic concepts of logic in an informal way so that at the end of the introduction, the first loop has reached the starting point with a clearer notion of what logic is. A second loop goes through the rest of the chapters of the book with a formal and rigorous approach to logic and with the same aim: give to the reader an ever clearer notion of logic.

We shall be concerned here only with the introduction. Frege's influence is patent in it and Church even declares that "the theory which will be adopted here, [is] due in its essentials to Gottlob Frege" [Church 1956, 4]. It had already been a while since Church had claimed to be Fregean, as his rather successful attempt to formalize Frege's theory of sense and denotation in 1951<sup>1</sup> shows. This introduction, being an informal introduction to logic, is readable by anybody: beginners in logic, experts or even people with a grudge against anything seeming too mathematical, and thus gives the reader a clear 68-page exposition of the theory of meaning underlying Church's usually more technical work.

The question is then: Is Church faithful to Frege? or: Are Frege's ideas so diluted and modified through Church's writing that Frege's influence can only be traced to mere inspiration? In other words, is Church's picture of Frege a good one?

To answer this question, we shall have to consider two separate things. The first is whether or not Church's account of the basic concepts of logic are Fregean. The second concerns to what extent the unprepared reader may believe that the account is Fregean. We shall therefore reflect on the four possible cases: (1) when Church adopts Frege's views and admits it, (2) when he does not adopt them and admits it, (3) when he does adopt them and does not say so, and (4) when he does not adopt them and does not say so.

We will follow Church's argumentation, which can be divided into two main parts exposing basic concepts of logic, with the first part being on the theory of meaning of proper names: where they stand in logic, what their sense and denotation are and how they are related to variables. The second part concerns functions which are considered first in relation with proper names, then with concepts and finally with sentences. As sentences are regarded as names, what is said in the first part can apply to sentences and so does not have to be repeated, but we shall have to deal with the theory stating that sentences are names. In a third and final part we shall follow Church as he takes a step back and looks at logic in general and examines the different methods in logic.

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1. See [Church 1951], though the project goes back at least to [Church 1946] and [Church 1942].

## 2 Proper names

### 2.1 Natural language and proper names

Church opens his introduction on a brief insight on what logic is. We shall come back to this in the last part, when we shall assess Church's presentation of logic in the whole introduction. He then starts his tour of the basic concepts in logic with language, in which the first expression Church takes is the proper name:

One kind of expression which is familiar in the natural languages, and which we shall carry over also to formalized languages, is the *proper name*. [Church 1956, 3]

This way of presenting is very close to Frege's own introduction of proper names in "Über Sinn und Bedeutung":

Der Sinn eines Eigennamens wird von jedem erfaßt, der die Sprache oder das Ganze von Bezeichnungen hinreichend kennt, der er angehört. [Frege 1892, 27]<sup>2</sup>

Both stress the fact that proper names are essential in natural languages and so justify their coming first: everybody knows what their use is.

Church expounds on proper names to get closer to Frege's own notion. He thus defines proper names as the names "arbitrarily assigned to denote in a certain way", but also the names who have "a structure that expresses some analysis of the way in which they denote" [Church 1956, 3]. The key word here is obviously "denotation", Church's translation for the German "Bedeutung". This notion is very important in Frege's ontology and it lies at the core of the definition of proper names:

Aus dem Zusammenhange geht hervor, daß ich hier unter „Zeichen“ und „Namen“ irgendeine Bezeichnung verstanden habe, die einen Eigennamen vertritt, deren Bedeutung also ein bestimmter Gegenstand ist. [Frege 1892, 27]<sup>3</sup>

This notion of *Bedeutung* is central to the theory of meaning of proper names Church presents in this introduction and which admittedly comes from Frege's paper "Über Sinn und Bedeutung".

So as far as the basic ontology is concerned, Church stays very close to Frege and pronounces this filiation. Can as much be said about the theory based on this ontology?

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2. "The sense of a proper name is grasped by everybody who is sufficiently familiar with the language or totality of designations to which it belongs" [Frege 1948, 210].

3. "It is clear from the context that by 'sign' and 'name' I have here understood any designation representing a proper name, whose referent [*Bedeutung*] is thus a definite object" [Frege 1948, 210].

## 2.2 Sense and denotation

Once these notions have been introduced, they need to be explained. How does Church explain the notions of sense (*Sinn*) and of denotation (*Bedeutung*)?

Both words express meaning, but of two different kinds. As Church puts it “a proper name *denotes* or *names* that of which it is a name” [Church 1956, 4] and “the thing denoted will be called the *denotation*” [Church 1956, 5], but “besides the denotation, we ascribe to every proper name another kind of meaning, the *sense* [...]. Roughly, the sense is what is grasped when one understands a name” [Church 1956, 6].

So the meaning of a proper name is divided into the object denoted by the name and the sense of the name. This exposition of sense and denotation seems to be faithful to Frege's ideas and the two of them even wrap up this distinction in a similar way, Frege writing:

Um einen kurzen und genauen Ausdruck möglich zu machen, mögen folgende Redewendungen festgesetzt werden:  
Ein Eigename (Wort, Zeichen, Zeichenverbindung, Ausdruck) drückt aus seinen Sinn, bedeutet oder bezeichnet seine Bedeutung, [Frege 1892, 31]<sup>4</sup>

and Church's own passage being almost a translation of Frege's:

We shall say that a name *denotes* or *names* its denotation and *expresses* its sense. [Church 1956, 5]

Still, one may question Church's choices of translation. Translations of key concepts can indeed be tricky, for one wants the word in the translated language to convey all the meaning of the translated word in its language and nothing more. Church obviously uses “sense” for the German “Sinn”. This is a translation very close to the German and is agreed upon. But “denotation” for “Bedeutung” is not an obvious answer<sup>5</sup> and may even skew Frege's theory in English.

According to Tugendhat, *Bedeutung* in German has the sense of *meaning*, but also of *significance* and *importance* [Tugendhat 1970]. Church makes the connexion between *denotation* and *meaning* clear when defining *denotation*, while completely leaving the *importance* aspect out.

Yet as Tugendhat points out, Frege uses *Bedeutung* in “Über Sinn und Bedeutung” in an unexpected way for the German reader and therefore arouses curiosity that will entice them to understand Frege's novel point of view. So

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4. “To make short and exact expressions possible, let the following phraseology be established:

A proper name (word, sign, sign combination, expression) *expresses* its sense, *refers to* or *designates* its referent” [Frege 1948, 214].

5. For example, Geach and Black [Frege 1952] have preferred to translate Frege's “Bedeutung” by “reference” rather than “denotation”.

by choosing an unsatisfactory translation, according to Tugendhat, “the translators have preferred to withhold from English readers the puzzlement which every German reader experiences with this word on first reading Frege’s essay ‘Über Sinn und Bedeutung’” [Tugendhat 1970, 177]. But the readers of Church are not the readers intended by Frege’s article. And if Church claims his theory to be Fregean, then the theory should probably use Frege’s concepts as they are in his theory, but it does not bind Church to follow Frege’s argumentation.

Church’s main concern is to produce a basis for formal logic whereas in “Über Sinn und Bedeutung” Frege is mostly investigating what identity is. So as long as *denotation* has the same place in Church’s theory as *Bedeutung* in Frege’s theory and Church does not need the additional meaning “Bedeutung” has in German, this translation choice cannot be held long against him.<sup>6</sup>

### 2.3 Getting closer to mathematics

So what does it mean to understand a name? It is knowing its sense. And what is understanding a language? It is, according to Church, knowing the senses of all the names in the language, “but not necessarily knowing which senses determine the same denotation, or even which senses determine denotations at all” [Church 1956, 7].

Frege’s main interest is not in natural languages but in a language with no ambiguity so that each name expresses only one sense and nothing is left to guesswork, as Mark Textor [Textor 2011] stresses in the chapter “What is a Begriffsschrift good for?”. If the content of Frege’s thought may have evolved between the writing of the *Begriffsschrift* and the writing of “Über Sinn und Bedeutung”, his ambition remains the same.

And Church shares this ambition with Frege since his book is an introduction to such a language, properly developed in the main body of the *Introduction to Mathematical Logic*:

In a well constructed language of course every name should have just one sense, and it is intended in the formalized languages to secure such univocacy. [Church 1956, 7]

Church expounds on a cause of the equivocity of natural languages and claims this analysis to be Fregean. Such a cause is the oblique use of names, which modifies the usual relation among the expression, its sense, and its reference:

the sense which the name would express in its ordinary use [becomes] the denotation when the name is used obliquely. [Church 1956, 8]

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6. *Denotation* was also the translation commonly used after Russell’s 1905 paper “On Denoting”.

This account of different uses of names seems to sum up Frege's view in "Über Sinn und Bedeutung":

Wenn man in der gewöhnlichen Weise Worte gebraucht, so ist das, wovon man sprechen will, deren Bedeutung. Es kann aber auch vorkommen, daß man von den Worten selbst oder von ihrem Sinne reden will. Jenes geschieht, z.B., wenn man die Worte eines anderen in gerader Rede anführt. Die eigenen Worte bedeuten dann zunächst die Worte des anderen, und erst diese haben die gewöhnliche Bedeutung. [Frege 1892, 28]<sup>7</sup>

Such a precise language as the one aimed should not allow the same name to have different meanings depending on the context. So an expression should always have the same sense and denotation and one should not be able to modify these without modifying the expression itself at the same time.

A language of this kind is reached by getting closer to mathematics. Having separated the form and the content of a language makes it easy for name expressions to be understood as signs just like in mathematics. Church introduces new terms: constants, variables and forms. Proper names having a denotation become constants, a variable "is a symbol whose meaning is like that of a proper name or constant except that the single denotation of the constant is replaced by the possibility of various *values* of the variable" and a form is what is "obtained from the complex name by [...] replacing one of the constituent names by a variable" [Church 1956, 9].

So at this point Church seems completely faithful to Frege. We have a goal: obtaining a language that enables us to express ourselves with no ambiguity and carry out gapless demonstrations. And we have made the first steps towards it: names are divided according to form and content with the sign of the name on one side and on the other the meaning of the name divided into its sense and its denotation. For the language to be as exact as possible, names must have only one sense and it must not change according to the context. And finally names are considered as constants or variables according to whether or not they have a fixed denotation.

But a language is not only about designating objects, it is also about identifying relations between things. This is where Frege's innovative use of functions comes up.

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7. "If words are used in the ordinary way, one intends to speak of their referents. It can also happen, however, that one wishes to talk about the words themselves or their sense. This happens, for instance, when the words of another are quoted. One's own words then first designate words of the other speaker, and only the latter have their usual referents" [Frege 1948, 211].

### 3 Functions and sentences

#### 3.1 Functions

As Frege does in *Begriffsschrift* [Frege 1879] and “Über Sinn und Bedeutung” [Frege 1892], Church informally introduces the notion of function by first defining what a function is in a mathematical sense and using mathematical examples. By stressing the roles of the argument and of the value:

By a *function* [...] we shall understand an operation which, when applied to something as *argument*, yields a certain thing as the *value* of the function *for* that argument. [Church 1956, 15],

he is, according to Anthony Kenny, very close to Frege:

Mathematicians, Frege says, sometimes say that when two variables  $x$  and  $y$  are correlated by a law then  $y$  is a function of  $x$ . But this is an unfortunate way of speaking. A better way to put the matter is that  $y$  is the value of a certain function for  $x$  as argument. Frege gives the name ‘the value of a function for an argument’ to the result of completing that function with the argument in question. [Kenny 1995, 104]

The function itself, according to Church, is some kind of process which lies in the “yielding or determination of a value from each argument in the range of the function” [Church 1956, 15].

To make the relation clear between functions, arguments and values, Church explains that “to denote the value of a function for a given argument, it is usual to write a name of the function, followed by a name of the argument between parentheses” [Church 1956, 16], or better to use blank spaces instead of the argument and the value and thus comes very close to Frege's own presentation of functions [Frege 1891, 6], [Frege 1952, 24]. But things start to be more complex as Church states that these blanks may be filled by constants, variables or *forms*.

Forms in themselves seem harmless enough and the fact that Church introduced a new term for a certain kind of expression did not seem worth noticing until they were used in functions. Forms are formed by replacing in a complex name one or more constituent names by a variable [Church 1956, 10]. This notion is not present in Frege and has been imported by Church from algebra [Church 1956, 10, n.26].

Frege's notion of unsaturatedness (*Ungesättigkeit*) can be defined as follows:

A sign is unsaturated if and only if it contains at least one empty place where it is possible to introduce another sign which has the effect of ‘completing’ the given initial expression. [Angelelli 1967, 173]



Forms and functions could be seen as unsaturated expressions: they both have values only when they are completed with respectively a value [Church 1956, 10] or an argument. Functions and forms would thus be easily distinguished from objects or at least names of objects (be they constants, sentences or variables). Functions and forms would be unsaturated expressions and names of objects would be saturated expressions. But Church explicitly rejects the use of unsaturatedness and claims his notion of function to be closer to Frege's "Werthverlauf einer Funktion" [Church 1956, 13, n. 32].

This rejection of unsaturatedness is made explicit in Church's "A Formulation of the Logic of Sense and Denotation" where he abandons "Frege's notion of a function [...] as something ungesättigt, in favor of a notion according to which the name of a function may be treated in the same manner as any other name, provided that distinctions of type are observed" [Church 1951, 4], the introduction of the type theory being an essential modification to Frege's theory, a "means of avoiding the logical antinomies" [Church 1951, 4].<sup>8</sup>

So here we have on one hand Frege who believes that concepts are one kind of functions and that expressions refer to objects or concepts according to whether they are saturated or not. On the other hand, Church introduces forms where Frege did not feel the need for them, does not accept the unsaturated account for functions and thinks concepts "as non-linguistic in character" and uses them in a different way than Frege [Church 1956, 6].

## 3.2 Truth-values and sentences

Frege's concepts are a certain kind of function. They are "functions whose value is always a truth-value" [Kenny 1995, 111]. There are only two truth-values for Frege, namely the True and the False. Truth-values are a consequence of Frege's wish to extend the notion of functions to encompass more than just mathematics:

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8. Type theory can help understanding where Church and Frege stand as far as functions and unsaturatedness are concerned [Sundholm 2012]. Frege's unsaturated expressions are dependent entities because they depend on what fills the hole in the expression. Forms and Church's functions are independent from the variables. Church states this when he introduces forms: "the value thus obtained for the form is independent of the choice of a particular name of the given value of  $x$ " [Church 1956, 9]. So forms do not need to be completed in the same way unsaturated expressions need to be saturated to be really something complete. As for functions, they are "an operation which, when applied to something as *argument*, yields a certain thing as the *value* of the function *for* that argument" [Church 1956, 15], they are something *applied* to something else and not a certain kind of substitution (in Frege's functions, applying the function means doing a substitution). This is why Church refers to Frege's courses-of-value: these are also independent from the variable and applied by an application function.

wir müssen weiter gehen und Gegenstände ohne Beschränkung als Funktionswerte zulassen. [Frege 1891, 17]<sup>9</sup>

So mathematical objects are not the only possible values for functions. The values are the denotation for the function for an argument. On this point Church agrees with Frege and states that

denotation of  $N = f(\text{sense of } N)$  for all names  $N$  for which there is a denotation. [Church 1956, 19]

But whereas Frege has functions range over an untyped universe with every object in it, Church specifies, without mentioning in the introduction to the *Introduction to Mathematical Logic*<sup>10</sup> this departure from Frege, that “it lies in the nature of any given functions to be applicable to certain things” [Church 1956, 15].

Having only one big universe made it easy for Frege to compare concepts: they are identical if they have the same values for the same arguments. This gives every concept an extension which “would be a series of pairs, with one member of each pair being a truth-value, and the other member being an object” [Kenny 1995, 111]. These pairs can be represented as a curve on a graph. We can thus have a list of application functions like  $\langle 0, f(0) \rangle$  where 0 is an object and  $f(0)$  is its truth-value ;  $\langle 1, f(1) \rangle$ ;  $\langle 2, f(2) \rangle$ ... but since the function ranges over all objects in the universe, we also have  $\langle \text{mars}, f(\text{mars}) \rangle$ ,  $\langle \text{tea}, f(\text{tea}) \rangle$  and so on for every object. But the values can be objects themselves, and therefore a function ranges over them also. Adding to his axioms (the V<sup>th</sup> in the *Grundgesetze der Arithmetik*) a function that yields directly an object as its value led to a paradox.

Church therefore has good reasons not to accept an untyped universe. But he does follow—and even defends—Frege's controversial understanding of sentences as names. According to Church a sentence is “an aggregation of words which makes complete sense or expresses a complete thought” [Church 1956, 23], and he restricts himself to the assertive use of sentences. His reasoning is a pragmatic one:

An important advantage of regarding sentences as names is that all the ideas and explanations of §§ 01–03 can then be taken over at once and applied to sentences, and related matters, as a special case. [Church 1956, 24]

And he adds that even if he chose not to follow Frege on that point, he would be doing a theory of the meaning of sentences so close to what he did for names that it would lengthen and confuse his exposition for nothing.

9. “Not merely numbers but objects in general, are now admissible” [Frege 1952, 31].

10. Though as we have seen in the section 2.1, he has made it clear in his paper “A Formulation of the Logic of Sense and Denotation” [Church 1951, 4].

But these arguments seem to be begging the question: if Church supposes that sentences are some kind of name, of course writing a theory of meaning for them will then be similar to a theory of meaning of names.

If at some point we could wonder why Church did not follow Frege, we may here enquire why he is loyal to Frege.

### 3.3 Sentences as names

If sentences are considered as a certain kind of name, what has been said about proper names should be applicable to sentences. We know what the sense and denotation of proper names are, but what are the sense and denotation of sentences? This is a question Frege addresses in “Über Sinn und Bedeutung”, [Frege 1892, 32], [Frege 1948, 214]. In this key passage Frege gives a method for comparing senses, denotations and sentences. This method is that of substituting parts of a sentence by other parts which have the same sense or the same denotation, believing, according to Leibniz’s Principle of the Identity of Indiscernibles, that the resulting sentence should have respectively the same sense or denotation. This method is based on compositionality: the meaning of the sentence depends on the meaning of its parts. By replacing a part with another with the same meaning (be it sense or denotation), the meaning of the whole does not change. But as Tugendhat points out, “Frege says that we are interested in the significance of any part of a sentence only insofar as we are interested in the truth-value of the sentence” and adds:

Is this not to say that the significance of the parts of sentences, and in particular of names, consists in their contribution to the truth-value of the sentences into which they may enter? [Tugendhat 1970, 180]

This leads us to wonder in which way Frege’s compositionality goes. Is it bottom-up with the parts giving meaning to the sentence or top-down and “take the significance of sentences as primary” [Tugendhat 1970, 180]? Church seems to be in favor of the top-down interpretation since he considers that sentences are the “unit of expression in the natural languages” [Church 1956, 23], but then Leibniz’s Principle should not be applicable to names in order to conclude anything of the sentences.

Frege’s argument is based on this principle of interchangeability. But without making a distinction between the two possible directions for compositionality, the argument seems to lose force. What is more, according to Tugendhat, Frege’s argument did not even prove what he intended since:

Frege thought even then that he was proving by means of the principle of interchangeability that the truth-values of sentences correspond to the objects of names, whereas in fact the principle of interchangeability can only prove that the objects of names

correspond to the truth-values of sentences. [Tugendhat 1970, 182]

Had Frege fully justified his theory, Church could just call on it without giving yet again a proof of it in his own exposition. But Frege does not seem to prove that sentences are a kind of name whose denotation is a truth-value. Since neither does Church, he is using unproved elements in his exposition, his only justification being that it is more convenient that way. So concerning sentences as names, Church seems to give a good picture of Frege by being faithful to his theory but also to his way of argumenting it, which in this case is a pity.

So Church seems to have taken some liberties from Frege in this section. He does not accept Frege's distinction of functions and objects as unsaturated expressions and saturated ones, adds forms without warning they are not Fregean and rejects a universal domain without any mention of Frege. Since in many places Church explicitly states if he is or is not following Frege's views, and introduces these modifications here without warning, an unprepared reader could mistake them as being Fregean. What is more, when Church does actually fully follow Frege's theory, announces it and even defends it, it is a controversial part of the theory that has not been well argued by Frege and not convincingly defended by Church. His global conception of functions is nonetheless not that far from Frege's and even if his undertaking of Frege's theory of meaning of sentences needs a better justification, Church is being very faithful to Frege and presents well his theory.

The next passages ([Church 1956, §05 and §06]) in the introduction to the *Introduction to Mathematical Logic* concern mostly the introduction of connectives and quantifiers. Since Church has used the modern notation, we shall not delve into these paragraphs but go straight to Church's account of the methods in logic.

## 4 Logic and its methods

### 4.1 Different perspectives on logic

Up till now, our concern mostly consisted in pointing out in which respects Church differed from Frege. This was because Church claimed his theory was essentially due to Frege. Although Church does not warn the reader that he is not following Frege as closely anymore, he does use concepts from Hilbert, Tarski and Carnap [Church 1956, 47, n. 220]. This shift in position reflects the general shift of perspective in logic at that time.

There are two ways of looking at classical logic. The traditional perspective (from Aristotle to Frege) is contentful: logic is seen as a tool to say things. The modern perspective, following the practice of Tarski, Quine and Carnap,

sees logic as something to talk about, it becomes the object of study. This shift would be due to the fall of logicism: Frege's system, which is contentful and follows classical logic, was found inconsistent and all the subsequent attempts to found such a system failed. According to Sundholm, anybody wanting to keep classical logic had therefore no other choice but to drop the contentful aspect of logic [Sundholm 1998]. Church referring to the main instigators of the metamathematical shift gives good ground for considering him as part of the second trend. What is more, when he says that

our interest in formalized languages [is] less often in their actual and practical use as languages than in the general theory of such use and its possibilities in principle, [Church 1956, 47]

he is basically voicing the principle behind the metamathematical point of view.

Once this structural divide has been pointed out, the task which remains is not to search for the differences between Frege's and Church's theories, but rather to understand how much of Frege's views still survive in Church's theory. This should enable us to assess Church's general picture of Frege despite the structural discrepancies.

## 4.2 The logistic method

Church points out two different methods in logic: the axiomatic method and the logistic method. He states that the axiomatic method is usually used in mathematics and does not intend to set up a formalized language. It is used for example in theoretical syntax [Church 1956, 59]. The logistic method, on the other hand is "the method of setting up a formalized language" [Church 1956, 56]. According to Rouilhan, Church's logicist method is nothing but Frege's own method [Rouilhan 1988, 15]. This does not contradict what has just been said: sharing the same method does not mean that method has to be applied to the same conception of the role of language in logic.

Still according to Rouilhan, the core of this method would be the "idéal de démonstrativité" [Rouilhan 1988, 16] which Frege shares in his ambition to keep intuition at bay and have gap-free proofs. This is the reason why he needed an ideography: as we have already mentioned, he wanted a language that could be extremely precise in what it said and that could express everything that is needed in inferences:

[Die Begriffsschrift] soll also zunächst dazu dienen, die Bündigkeit einer Schlusskette auf die sicherste Weise zu prüfen und jede Voraussetzung, die sich unbemerkt einschleichen will, anzuzeigen,

damit letztere auf ihren Ursprung untersucht werden könne. [Frege 1879, iv]<sup>11</sup>

This is exactly what Church is trying to do with his logistic system and in this respect it is obvious how much modern logic is indebted to Frege. In the formal logic Church is building, one must be able to determine, for everything expressed, what that thing is: a constant, a well-formed formula, a proof or simply nonsense. This is why Church insists so much on effectiveness [Church 1956, 50].

So as far as the method goes, Church follows Frege very closely. The logistic method is to set up a formalized language, but for Church what has been set up is only a purely syntactical part devoid of meaning.

### 4.3 What is logic?

In order to have a formalized language, Church points out that what has been done up to now is not enough:

After setting up the logistic system as described, we still do not have a formalized language until an *interpretation* is provided. [Church 1956, 54]

It is from that point on that Church does not follow Frege for the reasons already mentioned. The separation between the logistic system and its interpretation makes it necessary for Church to test the soundness of the system. This is something that Frege did not find necessary for he regarded axioms in a traditional sense: as truths. And since his rules of inference preserved truth, soundness was not a problem.

Giving an interpretation means giving semantical rules that “prescribe for every well-formed formula either how it denotes [...] or else how it has values” [Church 1956, 54]. Semantics are the formalized meta-language used to state the semantical rules. So compared to the initial object-language, semantics are a meta-meta-language. This division of language into levels and meta-levels is not present in Frege, though he does have the distinction between the sign of a name, its sense and its denotation, which is reminiscent at a local level of the link that can be made between pure syntax, an interpretation of it and what can actually be found in the world. In this sense, the tradition initiated by Tarski, Quine and Carnap that Church follows seems to be a generalization of Frege's trichotomy and its application to languages themselves.

In the first paragraph (§00), Church had only examples to resort to in order to explain his view of formal logic:

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11. “Its first purpose, therefore, is to provide us with the most reliable test of the validity of a chain of inferences and to point out every presupposition that tries to sneak in unnoticed” [Frege 1967, 6].

Traditionally, (formal) logic is concerned with the analysis of sentences or of propositions and of proof with attention to the *form* in abstraction from the *matter*. This distinction between form and matter is not easy to make precise immediately, but it may be illustrated by examples. [Church 1956, 1]

Presently, the reader is supposed to have grasped most of the content of the theory and should be able to follow Church as he starts his presentation of logic over again, but in a rigorous and formal way.

## Conclusion

So Church gives a good broad picture of Frege, but many precise aspects differ. These differences could easily be explained by stressing the fact that he is only using Frege to feed his own theory, not making a dissertation on Frege. Still, Church informs the reader that his theory is similar to that of Frege's and warns at some times when he departs from Frege. From these informations, an unprepared reader may reasonably believe that all the other aspects of Church's are Fregean, and we have seen that some are and others are not. We may therefore say that as a general introduction, Church's picture of Frege is a good one but it should not be a reference when reading Frege in detail.

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